Intrinsic conformal geometry of gravitational waves at Null-Infinity

Yannick Herfray

(Université Libre de Bruxelles)

Overview and motivations

- I will present a geometrical framework which, I believe, is the most adequate when dealing with null-infinity
- It generalises Tractor calculus from conformal geometry. In particular, it is by construction manifestly conformally invariant.
- It gives a natural and satisfying answer to an old question:

  What is the geometrical (i.e invariant) structure induced at null-infinity by the presence of gravitational waves?

  This is a choice of tractor connection.
- It also extends previous works on "Carroll geometry" (gives a definition of "strong conformal Carroll structure").
- This is very likely to be the correct formalism to efficiently couple fields to the "Carrollian field theory" at null-infinity.

Asymptotically flat space-times and gravitational waves

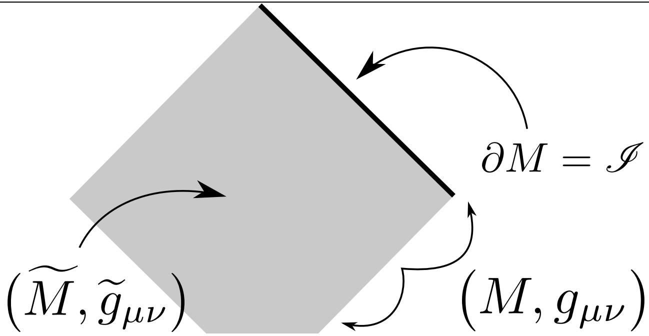
Asymptotically flat space-times

The space-time $\left(\widetilde{M}, \tilde{g}_{\mu\nu}\right)$ is **asymptotically simple** if there exists a space-time $(M, g_{\mu\nu})$ with boundary $\partial M = \mathscr{I}$ such that

- $\widetilde{M}$ is diffeomorphic to the interior $M \backslash \mathscr{I}$ of M
- there exists $\Omega \in C^\infty(M)$ a boundary defining function for $\mathscr{I}$ i.e

$$\Omega > 0 \text{ on } M, \qquad \Omega = 0,\ d\Omega \neq 0 \text{ on } \mathscr{I}$$

- $\tilde{g}_{\mu\nu} = \dfrac{1}{\Omega^2} g_{\mu\nu}$ on $\widetilde{M}$



The space-time $\left(\widetilde{M}, \tilde{g}_{\mu\nu}\right)$ is **asymptotically simple** if there exists a space-time $(M, g_{\mu\nu})$ with boundary $\partial M = \mathscr{I}$ such that

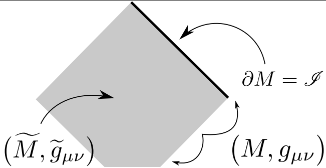
- $\widetilde{M}$ is diffeomorphic to the interior $M \backslash \mathscr{I}$ of M
- there exists $\Omega \in C^{\infty}(M)$ a boundary defining function for $\mathscr{I}$ i.e

$$\Omega > 0 \text{ on } M, \qquad \Omega = 0,\ d\Omega \neq 0 \text{ on } \mathscr{I}$$

- $\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}$ on $\widetilde{M}$

It is **asymptotically <u>flat</u> (resp AdS/dS)** if on top of this

- $\tilde{g}_{\mu\nu}$ is Einstein
- $g^{\mu\nu}\left(d\Omega_\mu, d\Omega_\nu\right) = \underline{0}$ (resp ± 1) on $\mathscr{I}$



The space-time $\left(\widetilde{M}, \tilde{g}_{\mu\nu}\right)$ is **asymptotically simple** if there exists a space-time $(M, g_{\mu\nu})$ with boundary $\partial M = \mathscr{I}$ such that

- $\widetilde{M}$ is diffeomorphic to the interior $M \backslash \mathscr{I}$ of M
- there exists $\Omega \in C^{\infty}(M)$ a boundary defining function for $\mathscr{I}$ i.e

$$\Omega > 0 \text{ on } M, \qquad \Omega = 0,\ d\Omega \neq 0 \text{ on } \mathscr{I}$$

- $\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}$ on $\widetilde{M}$

It is **asymptotically <u>flat</u> (resp AdS/dS)** if on top of this

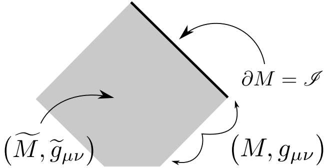
- $\tilde{g}_{\mu\nu}$ is Einstein
- $g^{\mu\nu}\left(d\Omega_{\mu}, d\Omega_{\nu}\right) = \underline{0}$ (resp ± 1) on $\mathscr{I}$

⚠ There is nothing unique about Ω nor $g_{\mu\nu}$! Rather one is working with an equivalence class:

$$\left(g_{\mu\nu}, \Omega\right) \sim \left(\lambda^2 g_{\mu\nu}, \lambda\Omega\right) \qquad \lambda \in C^{\infty}(M)$$

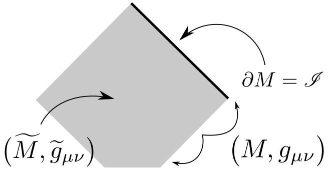
"Weak" null-infinity structure

Let $([g_{\mu\nu}], [\Omega], M)$ be an asymptotically flat space-time (in particular $\frac{1}{\Omega^2} g_{\mu\nu}$ is Einstein).



"Weak" null-infinity structure

Let $([g_{\mu\nu}], [\Omega], M)$ be an asymptotically flat space-time (in particular $\frac{1}{\Omega^2} g_{\mu\nu}$ is Einstein).



The "weak null-infinity structure" induced on the boundary $\mathscr{I}$ is

- a degenerate conformal metric $[h_{ab} \sim \lambda^2 h_{ab}]$ with one-dimensional kernel, obtained as
$$h_{ab} := g_{\mu\nu}\big|_{\mathscr{I}}$$
- an equivalence class of vector fields $\big[(n^a, h_{ab}) \sim (\lambda^{-1} n^a, \lambda^2 h_{ab})\big]$, obtained as
$$n^a := g^{\mu\nu} d\Omega_\nu\big|_{\mathscr{I}}$$
- with compatibility conditions $n^a h_{ab} = 0$ (following from $g^{\mu\nu} d\Omega_\mu d\Omega_\nu = 0$) and $\mathcal{L}_n h_{ab} \propto h_{ab}$ (following from Einstein equations).

"Universal" null-infinity structure

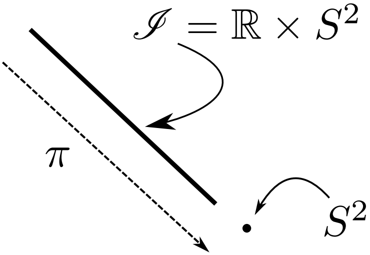
Let $\mathscr{I}$ be 3-dimensional manifold, we will say that it is equipped with the **universal null-infinity structure** if

- $\mathscr{I} = S^2 \times \mathbb{R}$ is the total space of a fibre bundle $\mathscr{I} \xrightarrow{\pi} S^2$

it is equipped with

- the conformal-sphere metric $[h^{(S^2)}_{AB}]$ on S^2
- an equivalence class $[n^a]$ of vertical vector fields $n^a d\pi_a = 0$

NB: then $h_{ab} = \pi^* h^{(S^2)}_{AB}$ automatically implies $n^a h_{ab} = 0$, $\mathcal{L}_n h_{ab} = h_{ab}$.



“Universal” null-infinity structure

Let $\mathscr{I}$ be 3-dimensional manifold, we will say that it is equipped with the **universal null-infinity structure** if

- $\mathscr{I} = S^2 \times \mathbb{R}$ is the total space of a fibre bundle $\mathscr{I} \xrightarrow{\pi} S^2$

it is equipped with

- the conformal-sphere metric $[h^{(S^2)}_{AB}]$ on S^2
- an equivalence class $[n^a]$ of vertical vector fields $n^a d\pi_a = 0$

NB: then $h_{ab} = \pi^* h^{(S^2)}_{AB}$ automatically implies $n^a h_{ab} = 0$, $\mathcal{L}_n h_{ab} = h_{ab}$.

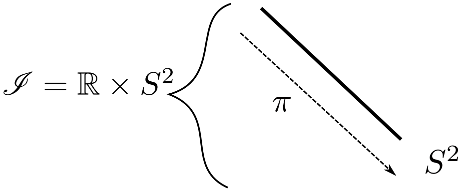
Symmetry group

The group of diffeomorphism of $\mathscr{I}$ preserving the universal null-infinity structure is the BMS group:

$$BMS\,(4) = \mathcal{C}^\infty(S^2) \rtimes SO\,(3,1)$$

Coordinates

Let $\left(\mathscr{I} \to S^2, [h_{ab}^{(S^2)}], [n^a]\right)$ be a manifold equipped with the universal null-infinity structure.

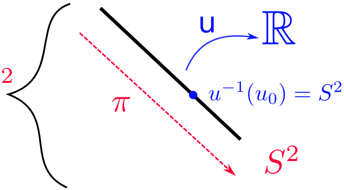


A **well-adapted trivialisation** (u, h_{AB}) is a choice

Coordinates

Let $\left(\mathscr{I} \to S^2, [h_{ab}^{(S^2)}], [n^a]\right)$ be a manifold equipped with the universal null-infinity structure.

$\mathscr{I} = \mathbb{R} \times S^2$



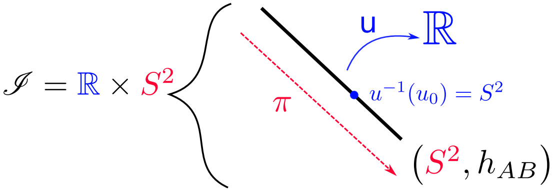
A **well-adapted trivialisation** (u, h_{AB}) is a choice

- of trivialisation $u : \mathscr{I} \to \mathbb{R}$ of $\mathscr{I} \xrightarrow{\pi} S^2$

$$(u, \pi) : \left| \begin{array}{ccc} \mathscr{I} & \to & \mathbb{R} \times S^2 \\ x & \mapsto & (u(x), \pi(x)) \end{array} \right.$$

Coordinates

Let $\left(\mathscr{I} \to S^2, [h_{ab}^{(S^2)}], [n^a]\right)$ be a manifold equipped with the universal null-infinity structure.



A **well-adapted trivialisation** (u, h_{AB}) is a choice

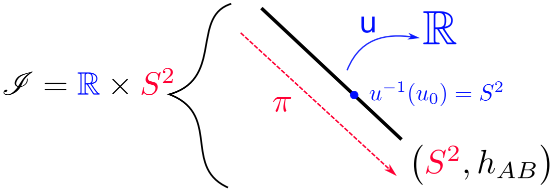
- of trivialisation $u : \mathscr{I} \to \mathbb{R}$ of $\mathscr{I} \xrightarrow{\pi} S^2$

$$(u, \pi) : \left| \begin{array}{ccc} \mathscr{I} & \to & \mathbb{R} \times S^2 \\ x & \mapsto & (u(x), \pi(x)) \end{array} \right.$$

- of representative $h_{AB} \in [h_{AB}^{(S^2)}]$
  (since $(n^a, h_{ab}) \sim \left(\lambda n^a, \lambda^2 h_{ab}\right)$, this also gives a representative $n^a \in [n^a]$)

Coordinates

Let $\left(\mathscr{I} \to S^2, [h_{ab}^{(S^2)}], [n^a]\right)$ be a manifold equipped with the universal null-infinity structure.



A **well-adapted trivialisation** (u, h_{AB}) is a choice

- of trivialisation $u : \mathscr{I} \to \mathbb{R}$ of $\mathscr{I} \xrightarrow{\pi} S^2$

$$(u, \pi) : \left| \begin{array}{ccc} \mathscr{I} & \to & \mathbb{R} \times S^2 \\ x & \mapsto & (u(x), \pi(x)) \end{array} \right.$$

- of representative $h_{AB} \in [h_{AB}^{(S^2)}]$
(since $(n^a, h_{ab}) \sim \left(\lambda n^a, \lambda^2 h_{ab}\right)$, this also gives a representative $n^a \in [n^a]$)
- with compatibility condition $n^a du_a = 1$ (i.e "$n^a = \partial_u$")

BMS coordinates

Let $(M, [g_{\mu\nu}], [\Omega])$ be an asymptotically flat space-times such that the induced structure on the boundary $\left(\mathscr{I}, [h_{ab}^{(S^2)}], [n^a]\right)$ is the universal null-infinity structure.

BMS coordinates

Choices of well-adapted trivialisation (u, h_{AB}) on $(\mathscr{I}, [h_{ab}], [n^a])$ are in one-to-one correspondence with BMS-coordinates on M i.e local coordinates

$$(u, \Omega, \pi) \left| \begin{array}{ccc} M & \to & \mathbb{R} \times \mathbb{R} \times S^2 \\ x & \to & (u(x), \Omega(x), y^A(x)) \end{array} \right.$$

on a neighbourhood of $\mathscr{I}$ in M such that

$$\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2} \left(2d\Omega du + h_{AB}(y) + \Omega C_{AB}(u, y) + \mathcal{O}\left(\Omega^2\right) \right)$$

Asymptotic shear and gravitational waves

BMS coordinates

Well-adapted trivialisations (u, h_{AB}) on $(\mathscr{I}, [h_{ab}], [n^a])$ are in one-to-one correspondence with BMS-coordinates on $(M, [g_{\mu\nu}], [\Omega])$

$$\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2}\left(2d\Omega du + h_{AB}(y) + \Omega C_{AB}(u, y) + \mathcal{O}\left(\Omega^2\right)\right)$$

The "asymptotic shear" C_{AB} is known to encode the gravitational radiation reaching null-infinity.

Asymptotic shear and gravitational waves

BMS coordinates

Well-adapted trivialisations (u, h_{AB}) on $(\mathscr{I}, [h_{ab}], [n^a])$ are in one-to-one correspondence with BMS-coordinates on $(M, [g_{\mu\nu}], [\Omega])$

$$\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2}\left(2d\Omega du + h_{AB}(y) + \Omega C_{AB}(u,y) + \mathcal{O}\left(\Omega^2\right)\right)$$

The “asymptotic shear” C_{AB} is known to encode the gravitational radiation reaching null-infinity.
$\Rightarrow$ this is however nothing like a tensor on $\mathscr{I}$!

Asymptotic shear and gravitational waves

BMS coordinates

Well-adapted trivialisations (u, h_{AB}) on $(\mathscr{I}, [h_{ab}], [n^a])$ are in one-to-one correspondence with BMS-coordinates on $(M, [g_{\mu\nu}], [\Omega])$

$$\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2}\left(2d\Omega du + h_{AB}(y) + \Omega C_{AB}(u,y) + \mathcal{O}\left(\Omega^2\right)\right)$$

The "asymptotic shear" C_{AB} is known to encode the gravitational radiation reaching null-infinity.
$\Rightarrow$ this is however nothing like a tensor on $\mathscr{I}$!

Had we chosen another well-adapted trivialisation
$\left(\hat{u} = \lambda\left(u - \xi\right), \hat{h}_{AB} = \lambda^2 h_{AB}\right)$ on $(\mathscr{I}, [h_{ab}], [n^a])$ with $\xi, \lambda \in \mathcal{C}^\infty\left(S^2\right)$
we would have

$$\begin{aligned} h_{AB} &\mapsto \hat{h}_{AB} = \lambda^2 h_{AB} \\ n^a &\mapsto \hat{n}^a = \lambda^{-1} n^a \\ C_{AB} &\mapsto \hat{C}_{AB} = \lambda C_{AB} - 2\left(\nabla_A \nabla_B\big|_0 \xi + \hat{u}\lambda\, \nabla_A\nabla_B\big|_0 \lambda^{-1}\right) \end{aligned}$$

Asymptotic shear and gravitational waves

BMS coordinates

Well-adapted trivialisations (u, h_{AB}) on $(\mathscr{I}, [h_{ab}], [n^a])$ are in one-to-one correspondence with BMS-coordinates on $(M, [g_{\mu\nu}], [\Omega])$

$$\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2}\left(2d\Omega du + h_{AB}(y) + \Omega C_{AB}(u, y) + \mathcal{O}\left(\Omega^2\right)\right)$$

The "asymptotic shear" C_{AB} is known to encode the gravitational radiation reaching null-infinity.

$\Rightarrow$ this is however nothing like a tensor on $\mathscr{I}$!

What is the (invariant) geometrical objects whose coordinates transform as the asymptotic shear?

Brief answer

The "asymptotic shear" C_{AB} parametrizes
a choice of "tractor connection" on $(M, [h_{AB}], [n^a])$.

Brief answer

The “asymptotic shear” C_{AB} parametrizes
a choice of “tractor connection” on $(M, [h_{AB}], [n^a])$.

More precisely...

Brief answer

The "asymptotic shear" C_{AB} parametrizes
a choice of "tractor connection" on $(M, [h_{AB}], [n^a])$.

More precisely...

- the tractor bundle is a natural vector bundle canonically associated to conformal manifolds (here needs to be adapted to *degenerate* conformal manifolds)

Brief answer

The "asymptotic shear" C_{AB} parametrizes
a choice of "tractor connection" on $(M, [h_{AB}], [n^a])$.

More precisely...

- the tractor bundle is a natural vector bundle canonically associated to conformal manifolds (here needs to be adapted to *degenerate* conformal manifolds)
- in conformal geometry the "normal" connection on this bundle is unique (for n$\geq$3)

Brief answer

The “asymptotic shear” C_{AB} parametrizes
a choice of “tractor connection” on $(M, [h_{AB}], [n^a])$.

More precisely...

- the tractor bundle is a natural vector bundle canonically associated to conformal manifolds (here needs to be adapted to *degenerate* conformal manifolds)
- in conformal geometry the “normal” connection on this bundle is unique (for n$\geq$3)
- however, for the degenerate conformal geometry of null-infinity, “null-normal” connections on the tractor bundle are not unique

Brief answer

The "asymptotic shear" C_{AB} parametrizes
a choice of "tractor connection" on $(M, [h_{AB}], [n^a])$.

More precisely...

- the tractor bundle is a natural vector bundle canonically associated to conformal manifolds (here needs to be adapted to *degenerate* conformal manifolds)
- in conformal geometry the "normal" connection on this bundle is unique (for n$\geq$3)
- however, for the degenerate conformal geometry of null-infinity, "null-normal" connections on the tractor bundle are not unique
- rather these null-normal tractor connections form an affine space modelled on trace-free symmetric tensor on S^2 (i.e "C_{AB}")

Brief answer

The “asymptotic shear” C_{AB} parametrizes
a choice of “tractor connection” on $(M, [h_{AB}], [n^a])$.

More precisely...

- the tractor bundle is a natural vector bundle canonically associated to conformal manifolds (here needs to be adapted to *degenerate* conformal manifolds)
- in conformal geometry the “normal” connection on this bundle is unique (for n$\geq$3)
- however, for the degenerate conformal geometry of null-infinity, “null-normal” connections on the tractor bundle are not unique
- rather these null-normal tractor connections form an affine space modelled on trace-free symmetric tensor on S^2 (i.e “C_{AB}”)
- **this is an invariant description** but choices of well-adapted trivialisation (u, h_{AB}) (equivalently BMS coordinates) acts as a trivialisation for this bundle, the tractor connection is then explicitly parametrized as a function of C_{AB}

Gravitational radiation as a “gauge” connection at null-infinity

What kind of object is this tractor connection ?

Brief answer,

The tractor connection is a “gauge” connection
for the Poincaré group $\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

What kind of object is this tractor connection ?

Brief answer,

> The tractor connection is a "gauge" connection
> for the Poincaré group $\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

In a well-adapted trivialisation (u, h_{AB}) we have

$$D_b = d_b + \begin{pmatrix} 0 & -\theta_{bC} & 0 & 0 \\ -\xi_b{}^A & \Gamma_b{}^A{}_C & \theta_b{}^A & 0 \\ 0 & \xi_{bC} & 0 & 0 \\ -\psi_b & -\frac{1}{2}C_{bC} & du_b & 0 \end{pmatrix} \in \mathbb{R}^4 \rtimes SO(3,1)$$

with

$$C_{bA} = C_{AB}\, \theta_b^B,$$

$$\xi_{bA} = \left(\frac{1}{2}\partial_u C_{AB} - \frac{R}{4} h_{AB}\right) \theta_b^B,$$

$$\psi_b = \frac{1}{4} R\, du_b - \frac{1}{2} \nabla^C C_{BC}\, \theta_b^B.$$

What kind of object is this tractor connection ?

Brief answer,

> The tractor connection is a "gauge" connection
> for the Poincaré group $\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

In a well-adapted trivialisation (u, h_{AB}) we have

$$D_b = d_b + \begin{pmatrix} 0 & -\theta_{bC} & 0 & 0 \\ -\xi_b{}^A & \Gamma_b{}^A{}_C & \theta_b{}^A & 0 \\ 0 & \xi_{bC} & 0 & 0 \\ \boxed{-\psi_b \quad -\frac{1}{2}C_{bC} \quad du_b} & & & 0 \end{pmatrix} \in \mathbb{R}^4 \rtimes SO(3,1)$$

with

$$C_{bA} = C_{AB}\, \theta_b^B,$$

$$\xi_{bA} = \left(\frac{1}{2}\partial_u C_{AB} - \frac{R}{4} h_{AB}\right) \theta_b^B,$$

$$\psi_b = \frac{1}{4} R\, du_b - \frac{1}{2}\nabla^C C_{BC}\, \theta_b^B.$$

What kind of object is this tractor connection ?

Brief answer,

> The tractor connection is a “gauge” connection
> for the Poincaré group $\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

In a well-adapted trivialisation (u, h_{AB}) we have

$$D_b = d_b + \begin{pmatrix} 0 & -\theta_{bC} & 0 & 0 \\ -\xi_b{}^A & \Gamma_b{}^A{}_C & \theta_b{}^A & 0 \\ 0 & \xi_{bC} & 0 & 0 \\ -\psi_b & -\frac{1}{2}C_{bC} & du_b & 0 \end{pmatrix} \in \mathbb{R}^4 \rtimes SO(3,1)$$

with

$$C_{bA} = C_{AB}\, \theta_b^B,$$

$$\xi_{bA} = \left(\frac{1}{2}\partial_u C_{AB} - \frac{R}{4} h_{AB}\right)\theta_b^B,$$

$$\psi_b = \frac{1}{4} R\, du_b - \frac{1}{2}\nabla^C C_{BC}\, \theta_b^B.$$

What kind of object is this tractor connection ?

Brief answer,

The tractor connection is a “gauge” connection
for the Poincaré group $\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

Why is the Poincaré group showing up here ?

What kind of object is this tractor connection ?

Brief answer,

The tractor connection is a "gauge" connection
for the Poincaré group $\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

Why is the Poincaré group showing up here ?

- We all know that Minkowski space $\mathbb{M}_4$ is an homogenous space for the Poincaré group,

$$\mathbb{M}_4 = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{SO}(3,1)}$$

What kind of object is this tractor connection ?

Brief answer,

The tractor connection is a “gauge” connection
for the Poincaré group $\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$

Why is the Poincaré group showing up here ?

- We all know that Minkowski space $\mathbb{M}_4$ is an homogenous space for the Poincaré group,

$$\mathbb{M}_4 = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{SO}(3,1)}$$

- A lesser known fact is that the conformal boundary $\mathscr{I}_{flat}$ of this homogeneous space is also an homogeneous space for the Poincaré group,

$$\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$$

- A lesser known fact is that the conformal boundary $\mathscr{I}_{flat}$ of this homogeneous space is also an homogeneous space for the Poincaré group,

$$\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$$

Wait a minute...

- A lesser known fact is that the conformal boundary $\mathscr{I}_{flat}$ of this homogeneous space is also an homogeneous space for the Poincaré group,

$$\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$$

Wait a minute...

we said that the symmetry group of the "universal null-infinity structure" $\left(\mathscr{I} \to S^2, [h^{(S^2)}_{AB}], [n^a]\right)$ is the (infinite dimensional) BMS group,

- A lesser known fact is that the conformal boundary $\mathscr{I}_{flat}$ of this homogeneous space is also an homogeneous space for the Poincaré group,

$$\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$$

Wait a minute...
we said that the symmetry group of the "universal null-infinity structure" $\left(\mathscr{I} \to S^2, [h^{(S^2)}_{AB}], [n^a]\right)$ is the (infinite dimensional) BMS group,

...now we are saying that the conformal boundary of Minkowski space $\mathscr{I}_{flat}$ comes with a preferred action of the Poincaré group.

How does these two facts stick together?

- A lesser known fact is that the conformal boundary $\mathscr{I}_{flat}$ of this homogeneous space is also an homogeneous space for the Poincaré group,

$$\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$$

Wait a minute...
we said that the symmetry group of the “universal null-infinity structure” $\left(\mathscr{I} \to S^2, [h^{(S^2)}_{AB}], [n^a]\right)$ is the (infinite dimensional) BMS group,

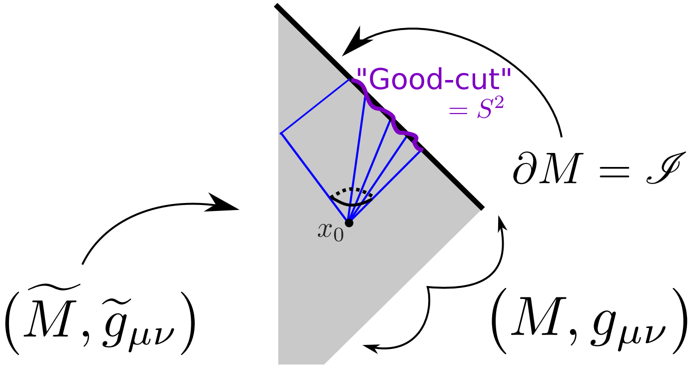
...now we are saying that the conformal boundary of Minkowski space $\mathscr{I}_{flat}$ comes with a preferred action of the Poincaré group.

How does these two facts stick together?

The conformal boundary of Minkowski space comes equipped with more than the universal structure, it is equipped with a set $\{s : S^2 \to \mathscr{I}\}_{s \in \mathcal{H}}$ of *good-cuts* on top of the universal structure

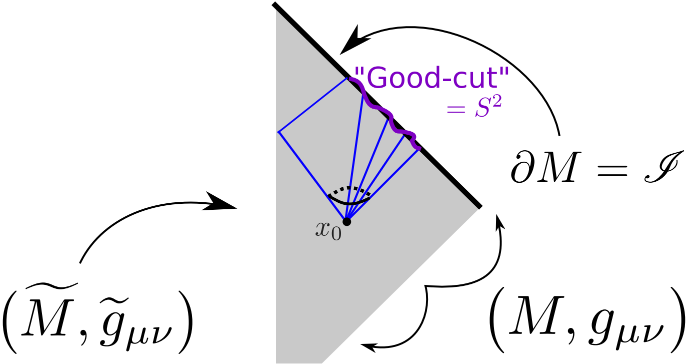
The conformal boundary $\mathscr{I}_{flat}$ of Minkowski space comes equipped with more than the universal structure, it is equipped with a set $\{s : S^2 \to \mathscr{I}\}_{s\in\mathcal{H}}$ of *good-cuts* on top of the universal structure

The conformal boundary $\mathscr{I}_{flat}$ of Minkowski space comes equipped with more than the universal structure, it is equipped with a set $\{s : S^2 \to \mathscr{I}\}_{s\in\mathcal{H}}$ of *good-cuts* on top of the universal structure



- Each of the null-cones emanating from $\mathbb{M}^4$ intersects $\mathscr{I}_{flat}$ along a "cut" $s\colon \mathscr{I} \to S^2$ (really the image of the section). (There is thus a 4-dimensional space $\mathcal{H}$ of these "good-cuts", $s \in \mathcal{H}$)

The conformal boundary $\mathscr{I}_{flat}$ of Minkowski space comes equipped with more than the universal structure, it is equipped with a set $\{s : S^2 \to \mathscr{I}\}_{s\in\mathcal{H}}$ of *good-cuts* on top of the universal structure



- Each of the null-cones emanating from $\mathbb{M}^4$ intersects $\mathscr{I}_{flat}$ along a "cut" $s\colon \mathscr{I} \to S^2$ (really the image of the section). (There is thus a 4-dimensional space $\mathcal{H}$ of these "good-cuts", $s \in \mathcal{H}$)
- The subgroup of BMS stabilizing these cuts is isomorphic to the Poincaré group

What kind of object is this tractor connection ?

Precise answer,

> the tractor connection is a Cartan connection
>
> modelled on the homogenous space $\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$

What kind of object is this tractor connection ?

Precise answer,

the tractor connection is a Cartan connection

modelled on the homogenous space $\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\diagup_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$

Flat tractor connections D...

What kind of object is this tractor connection ?

Precise answer,

> the tractor connection is a Cartan connection
>
> modelled on the homogenous space $\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$

Flat tractor connections D...

- Give an isomorphism $\phi \colon \mathscr{I} \to \mathscr{I}_{flat}$ to the homogenous space.

What kind of object is this tractor connection ?

Precise answer,

> the tractor connection is a Cartan connection
>
> modelled on the homogenous space $\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$

Flat tractor connections D...

- Give an isomorphism $\phi \colon \mathscr{I} \to \mathscr{I}_{flat}$ to the homogenous space.
- Defines a 4-dimensional space of good-cuts $\mathcal{H}_D$

What kind of object is this tractor connection ?

Precise answer,

> the tractor connection is a Cartan connection
>
> modelled on the homogenous space $\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$

Flat tractor connections D...

- Give an isomorphism $\phi \colon \mathscr{I} \to \mathscr{I}_{flat}$ to the homogenous space.
- Defines a 4-dimensional space of good-cuts $\mathcal{H}_D$
- Selects a copy of the Poincaré group inside the BMS group

What kind of object is this tractor connection ?

Precise answer,

> the tractor connection is a Cartan connection
>
> modelled on the homogenous space $\mathscr{I}_{flat} = {}^{\mathrm{Iso}(3,1)}\!\big/_{\mathrm{Carr}(3) \rtimes \mathbb{R}}$

Flat tractor connections D...

- Give an isomorphism $\phi \colon \mathscr{I} \to \mathscr{I}_{flat}$ to the homogenous space.
- Defines a 4-dimensional space of good-cuts $\mathcal{H}_D$
- Selects a copy of the Poincaré group inside the BMS group
- What is more,
  a good-cut then is equivalent to a covariantly constant section of the tractor bundle.

i.e

$$\left\{ s\colon S^2 \to \mathscr{I} \mid s \in \mathcal{H}_D \right\} \qquad \Leftrightarrow \qquad \left\{ \Phi^I \in \Gamma\left[\mathcal{T}\right] \mid D\Phi^I = 0 \right\}$$

Relation with asymptotically flat space-times

Let $(M, [g_{\mu\nu}], [\Omega])$ be an asymptotically flat space-times.

- $[\Omega]$ defines an “infinity tractor” $I^I \in \Gamma\left[\mathcal{T}_M\right]$.
- the sub-bundle $I^\perp\big|_{\mathscr{I}} \subset \mathcal{T}_M$ is canonically isomorphic to $\mathcal{T}_{\mathscr{I}}$
- the normal tractor connection of $[g_{\mu\nu}]$ induces on $\mathscr{I}$ a null-normal tractor connection
- the curvature of the tractor connection at $\mathscr{I}$ is parametrized by the unphysical Weyl tensor $\Omega^{-1}C_{\mu\nu\rho\sigma}$

A heuristic approach to the physics of null-infinity

Maxwell's equation on Minkowski space

Background: $(M = \mathbb{R}^4, g_{\mu\nu})$ where $g_{\mu\nu}$ is a flat metric.

Symmetry group: Poincaré group
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: 3+1 orthonormal splitting (t, x^i)
$\Rightarrow$ the Poincaré group sends a well-adapted set of coordinates to another.

Potential (in coordinates): (ϕ, A^i)

Field (in coordinates):
$$E^i = -(\nabla\phi)^i - \partial_t A^i$$
$$B^i = (\nabla \times A)^i$$

Field eqs (in coordinates):
$$\nabla . E = \rho$$
$$(\nabla \times B)^i - \partial_t E^i = j^i$$

Maxwell's equation on Minkowski space

Background: $(M = \mathbb{R}^4, g_{\mu\nu})$ where $g_{\mu\nu}$ is a flat metric.

Symmetry group: Poincaré group
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: 3+1 orthonormal splitting (t, x^i)
$\Rightarrow$ the Poincaré group sends a well-adapted set of coordinates to another.

Potential (in coordinates): (ϕ, A^i)

Field (in coordinates):
$$E^i = -(\nabla\phi)^i - \partial_t A^i$$
$$B^i = (\nabla \times A)^i$$

Field eqs (in coordinates):
$$\nabla.E = \rho$$
$$(\nabla \times B)^i - \partial_t E^i = j^i$$

$\Rightarrow$ Changing the set of adapted coordinates mixes the fields
$\Rightarrow$ This however preserve the "form" of Maxwell equations
$\Rightarrow$ If we fix a coordinate system, the Poincaré group takes solutions of the fields equations to others

Maxwell's equation on Minkowski space

Background: $(M = \mathbb{R}^4, g_{\mu\nu})$ where $g_{\mu\nu}$ is a flat metric.

Symmetry group: Poincaré group
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: 3+1 orthonormal splitting (t, x^i)
$\Rightarrow$ the Poincaré group sends a well-adapted set of coordinates to another.

Potential (invariant form) : a 1-form A_μ on M

Field (invariant form): $F_{\mu\nu} = (dA)_{\mu\nu}$

Field eqs (invariant form): $(d \star F)_{\mu\nu\rho} = J_{\mu\nu\rho}$

Maxwell's equation on Minkowski space

Background: $(M = \mathbb{R}^4, g_{\mu\nu})$ where $g_{\mu\nu}$ is a flat metric.

Symmetry group: Poincaré group
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: 3+1 orthonormal splitting (t, x^i)
$\Rightarrow$ the Poincaré group sends a well-adapted set of coordinates to another.

Potential (invariant form) : a 1-form A_μ on M

Field (invariant form): $F_{\mu\nu} = (dA)_{\mu\nu}$

Field eqs (invariant form): $(d \star F)_{\mu\nu\rho} = J_{\mu\nu\rho}$

$\Rightarrow$ This is a "Poincaré invariant" point of view (i.e does not depend on the choice of adapted coordinates)
$\Rightarrow$ The Poincaré group takes solutions of the fields equations to others
$\Rightarrow$ Gives a "4D-type" of intuition, allows to easily construct invariants, suggest Yang-Mills as generalisation, etc

Gravitational radiations at Null-infinity

Background: $\left(\mathscr{I} = \mathbb{R} \times S^2, [h_{AB}], [n^a]\right)$, i.e "universal null-infinity structure".

Symmetry group: BMS group, $BMS(3) = C^\infty\left(S^2\right) \rtimes \mathrm{SO}(3,1)$
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: (u, h_{AB})
$\Rightarrow$ the BMS group sends a well-adapted set of coordinates to another.

Potential (in coordinates): C_{AB}

Field (in coordinates): ψ_4,ψ_3,$Im\left(\psi_2\right)$

Field eqs (in coordinates): ψ_4,ψ_3,$Im\left(\psi_2\right)$ are choices of outcoming gravitational radiations

Gravitational radiations at Null-infinity

Background: $\left(\mathscr{I} = \mathbb{R} \times S^2, [h_{AB}], [n^a]\right)$, i.e "universal null-infinity structure".

Symmetry group: BMS group, $BMS(3) = C^\infty\left(S^2\right) \rtimes \mathrm{SO}(3,1)$
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: (u, h_{AB})
$\Rightarrow$ the BMS group sends a well-adapted set of coordinates to another.

Potential (in coordinates): C_{AB}

Field (in coordinates): ψ_4,ψ_3,$Im\left(\psi_2\right)$

Field eqs (in coordinates): ψ_4,ψ_3,$Im\left(\psi_2\right)$ are choices of outcoming gravitational radiations

$\Rightarrow$ Changing the set of adapted coordinates mixes the fields
$\Rightarrow$ This however preserve the “form” of the equations
$\Rightarrow$ If we fix a coordinate system, the BMS group takes solutions of the fields equations to others

Gravitational radiations at Null-infinity

Background: $\left(\mathscr{I} = \mathbb{R} \times S^2, [h_{AB}], [n^a]\right)$, i.e "universal null-infinity structure".

Symmetry group: BMS group, $BMS(3) = C^\infty\left(S^2\right) \rtimes \mathrm{SO}(3,1)$
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: (u, h_{AB})
$\Rightarrow$ the BMS group sends a well-adapted set of coordinates to another.

Potential (invariant form) : a tractor connection $D = d + A^I{}_J$ on M

Field (invariant form): $F^I{}_J = dA^I{}_J + A^I{}_K \wedge A^K{}_J$

Field eqs (invariant form): The curvature encodes the outcoming gravitational radiations "$F^I{}_J = J^I{}_J$"

Gravitational radiations at Null-infinity

Background: $\left(\mathscr{I} = \mathbb{R} \times S^2, [h_{AB}], [n^a]\right)$, i.e "universal null-infinity structure".

Symmetry group: BMS group, $BMS(3) = C^\infty\left(S^2\right) \rtimes \mathrm{SO}(3,1)$
(= subgroup of diffeomorphism preserving the background)

Well-adapted coordinates: (u, h_{AB})
$\Rightarrow$ the BMS group sends a well-adapted set of coordinates to another.

Potential (invariant form) : a tractor connection $D = d + A^I{}_J$ on M

Field (invariant form): $F^I{}_J = dA^I{}_J + A^I{}_K \wedge A^K{}_J$

Field eqs (invariant form): The curvature encodes the outcoming gravitational radiations "$F^I{}_J = J^I{}_J$"

$\Rightarrow$ This is a “BMS invariant” point of view (i.e does not depend on the choice of well-adapted coordinates)
$\Rightarrow$ The BMS group takes solutions of the fields equations to others
$\Rightarrow$ Gives a “conformally invariant” type of intuition, allows to easily construct invariants etc

Relations to Carroll manifolds (and others)

“Weak” Carroll geometries

weak Carroll structure $\sim$ universal null-infinity structure in a fixed scale

- $\mathscr{I} = \mathbb{R}^2 \times \mathbb{R}$ is the total space of a fibre bundle $\mathscr{I} \xrightarrow{\pi} \mathbb{R}^2$

it is equipped with

- the flat metric $h^{(flat)}_{AB}$ on $\mathbb{R}^2$
- a vertical vector fields n^a, $n^a d\pi_a = 0$

NB: then $h_{ab} = \pi^* h^{(S^2)}_{AB}$ automatically implies $n^a h_{ab} = 0$, $\mathcal{L}_n h_{ab} = h_{ab}$.

Symmetry group

The group of diffeomorphism of $\mathscr{I}$ preserving the weak Carroll structure is

$$Sym\left(\mathscr{I} \to \mathbb{R}^2, n^a, h^{(flat)}_{AB}\right) = \mathcal{C}^\infty(\mathbb{R}^2) \rtimes \mathrm{Iso}\,(2)$$

"Strong" Carroll geometries

Strong Carroll structure $\sim$ add an affine connection to the weak structure

- $\mathscr{I} = \mathbb{R}^2 \times \mathbb{R}$ is the total space of a fibre bundle $\mathscr{I} \xrightarrow{\pi} \mathbb{R}^2$

it is equipped with

- the flat metric $h_{AB}^{(flat)}$ on $\mathbb{R}^2$
- a vertical vector fields n^a, $n^a d\pi_a = 0$
- a compatible affine connection ∇ i.e $\nabla n^a = 0$, $\nabla h_{ab} = 0$

(With $h_{ab} = \pi^* h_{AB}$.)

Symmetry group

When ∇ is flat, the group of diffeomorphism of $\mathscr{I}$ preserving the strong Carroll structure is the Carroll group

$$Carr\,(3) = \mathbb{R}^3 \rtimes \mathrm{Iso}\,(2) \qquad \left(\subset C^\infty(\mathbb{R}^2) \rtimes \mathrm{Iso}\,(2)\right)$$

"Strong" conformal Carroll geometries

Strong conformal Carroll structure ?

- $\mathscr{I} = S^2 \times \mathbb{R}$ is the total space of a fibre bundle $\mathscr{I} \xrightarrow{\pi} S^2$

it is equipped with

- the flat metric $[h_{AB}^{(S^2)}]$ on S^2
- a vertical vector fields $[n^a]$, $n^a d\pi_a = 0$
- a compatible "null-normal" tractor connection D

Symmetry group

When D is flat, the group of diffeomorphism of $\mathscr{I}$ preserving the strong conformal Carroll structure is the Poincaré group

$$\mathrm{Iso}(3,1) = \mathbb{R}^4 \rtimes \mathrm{SO}\,(3,1) \qquad \big(\subset C^\infty(S^2) \rtimes \mathrm{SO}\,(3,1)\big)$$

Carroll vs Null-infinity

Since the "weak Carroll structure"

$$\left(\mathscr{I} \to \mathbb{R}^2, h^{(flat)}_{AB}, n^a\right)$$

is essentially a "weak Null-infinity structure"

$$\left(\mathscr{I} \to S^2, [h^{(S^2)}_{AB}], [n^a]\right)$$

together with a choice of flat representative

$$h^{(flat)}_{AB} \in [h_{AB}]$$

one might get the impression that working in a conformally invariant manner is unnecessary.

Carroll vs Null-infinity

Since the "weak Carroll structure"

$$\left(\mathscr{I} \to \mathbb{R}^2, h_{AB}^{(flat)}, n^a\right)$$

is essentially a "weak Null-infinity structure"

$$\left(\mathscr{I} \to S^2, [h_{AB}^{(S^2)}], [n^a]\right)$$

together with a choice of flat representative

$$h_{AB}^{(flat)} \in [h_{AB}]$$

one might get the impression that working in a conformally invariant manner is unnecessary.

In particular one might get the impression that "strong Carroll structures" (=affine connection ∇) are obtained from the "strong Null-infinity structure" (= tractor connection) by choosing a representative.

Carroll vs Null-infinity

Since the "weak Carroll structure"

$$\left(\mathscr{I} \to \mathbb{R}^2, h^{(flat)}_{AB}, n^a\right)$$

is essentially a "weak Null-infinity structure"

$$\left(\mathscr{I} \to S^2, [h^{(S^2)}_{AB}], [n^a]\right)$$

together with a choice of flat representative

$$h^{(flat)}_{AB} \in [h_{AB}]$$

one might get the impression that working in a conformally invariant manner is unnecessary.

In particular one might get the impression that "strong Carroll structures" (=affine connection ∇) are obtained from the "strong Null-infinity structure" (= tractor connection) by choosing a representative.

This is however not the case and indeed the asymptotic shear does not amounts to a choice of affine connection, even when working with a fixed representatives.

Carroll vs Null-infinity

In particular one might get the impression that "strong Carroll structures" (=affine connection ∇) are obtained from the "strong Null-infinity structure" (= tractor connection) by choosing a scale.

To convince oneself that it is wrong, it suffices to check that the subgroup

$$\mathbb{R}^4 \rtimes \mathrm{Iso}(2),$$

obtained as the subgroup of the Poincaré group $\mathbb{R}^4 \rtimes \mathrm{SO}(3,1)$ stabilizing the flat metric,

is <u>not</u> the Carroll group

$$Carr(3) = \mathbb{R}^3 \rtimes \mathrm{Iso}(2).$$

Therefore a tractor connection cannot be equivalent to an affine connection, even when working in a fixed scale.

Carroll vs Null-infinity

~~In particular one might get the impression that "strong Carroll structures" (=affine connection ∇) are obtained from the "strong Null-infinity structure" (= tractor connection D) by choosing a scale.~~

Carroll vs Null-infinity

~~In particular one might get the impression that "strong Carroll structures" (=affine connection ∇) are obtained from the "strong Null-infinity structure" (= tractor connection D) by choosing a scale.~~

Rather, when working with a fixed representative, a strong null-infinity structure D is equivalent to an equivalence class of affine connection :

$$\nabla \sim \hat{\nabla} \qquad \Leftrightarrow \qquad \nabla_a - \hat{\nabla}_a = f h_{ab} n^c \qquad \text{with} \qquad f \in C^\infty(\mathscr{I})$$

Carroll vs Null-infinity

~~In particular one might get the impression that "strong Carroll structures" (=affine connection ∇) are obtained from the "strong Null-infinity structure" (= tractor connection D) by choosing a scale.~~

Rather, when working with a fixed representative, a strong null-infinity structure D is equivalent to an equivalence class of affine connection :

$$\nabla \sim \hat{\nabla} \qquad \Leftrightarrow \qquad \nabla_a - \hat{\nabla}_a = f h_{ab} n^c \qquad \text{with} \qquad f \in C^\infty(\mathscr{I})$$

- These are the equivalence class of connections described by Ashtekar/Geroch: These are equivalent to choices of asymptotic shear.
- These were proposed as a geometrization of the asymptotic shear at null-infinity.

Comparison with Ashtekar/Geroch results

Ashetkar/Geroch connections

When working with a fixed representative, a strong null-infinity structure D is equivalent to an equivalence class of affine connection :

$$\nabla \sim \hat{\nabla} \qquad \Leftrightarrow \qquad \nabla_a - \hat{\nabla}_a = f h_{ab} n^c \qquad \text{with} \qquad f \in C^\infty(\mathscr{I})$$

Comparison with Ashtekar/Geroch results

Ashetkar/Geroch connections

When working with a fixed representative, a strong null-infinity structure D is equivalent to an equivalence class of affine connection :

$$\nabla \sim \hat{\nabla} \qquad \Leftrightarrow \qquad \nabla_a - \hat{\nabla}_a = f h_{ab} n^c \qquad \text{with} \qquad f \in C^\infty(\mathscr{I})$$

Even thought in principle equivalent to the tractor connection, in practice working with these equivalence class of connections is not very practical:

- How are we suppose to guess quantities invariant under this shift?
- Conformal invariance completely occulted

Comparison with Ashtekar/Geroch results

Ashetkar/Geroch connections

When working with a fixed representative, a strong null-infinity structure D is equivalent to an equivalence class of affine connection :

$$\nabla \sim \hat{\nabla} \qquad \Leftrightarrow \qquad \nabla_a - \hat{\nabla}_a = f h_{ab} n^c \qquad \text{with} \qquad f \in C^\infty(\mathscr{I})$$

Even thought in principle equivalent to the tractor connection, in practice working with these equivalence class of connections is not very practical:

- How are we suppose to guess quantities invariant under this shift?
- Conformal invariance completely occulted

On its side the tractor connection is

- a *bona fide* connection on the tractor bundle
  (one can construct invariants in the standard way)
- <u>and</u> is manifestly conformally invariant.

Gravity vacua

Gravity vaccua

The presence of gravitational wave at null-infinity is encoded in the curvature of the tractor connection.

The space Γ_0 of "gravity vaccua" is therefore the space of

flat null-normal tractor connections.

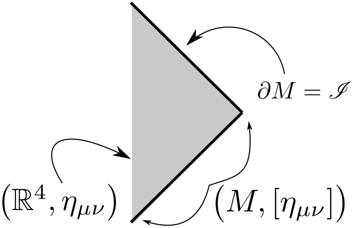
This space isn't a point, rather the BMS group act transitively on it with stabilisers isomorphic to the Poincaré group:

$$\Gamma_0 = {}^{BMS}\!\big/_{\mathrm{Iso}\,(3,1)}$$

Therefore the "gravity vacuum", Minkowski space, is not unique but rather we have a space of "gravity vacua" corresponding to all the possible flat null-normal tractor connections.

Wait...what do you mean Minkowski is not unique?

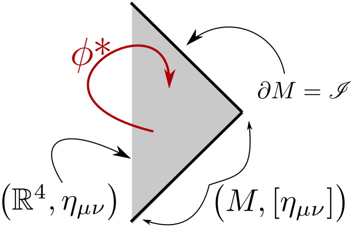
Let us consider a flat Lorentzian metric $\eta_{\mu\nu}$ which is conformally compact such that the conformal compactification M is Penrose's diamonds and the conformal boundary $\partial M = \mathscr{I}$ has a fixed universal null-infinity structure $(\mathscr{I} \to S^2, \boldsymbol{n}^a \boldsymbol{h}_{ab})$:



This is “a” Minkowski space-time. Is this unique?

Wait...what do you mean Minkowski is not unique?

Let us consider a flat Lorentzian metric $\eta_{\mu\nu}$ which is conformally compact such that the conformal compactification M is Penrose's diamonds and the conformal boundary $\partial M = \mathscr{I}$ has a fixed universal null-infinity structure $\left(\mathscr{I} \to S^2, \boldsymbol{n}^a \boldsymbol{h}_{ab}\right)$:

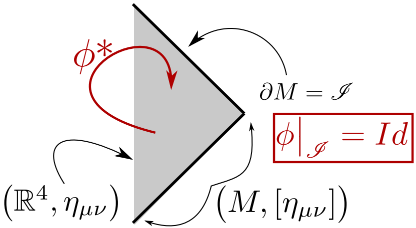


This is "a" Minkowski space-time. Is this unique?

- surely no for any diffeomorphism ϕ will send a solution to another $\phi^* \eta$.

Wait...what do you mean Minkowski is not unique?

Let us consider a flat Lorentzian metric $\eta_{\mu\nu}$ which is conformally compact such that the conformal compactification M is Penrose's diamonds and the conformal boundary $\partial M = \mathscr{I}$ has a fixed universal null-infinity structure $(\mathscr{I} \to S^2, \boldsymbol{n}^a \boldsymbol{h}_{ab})$:



This is "a" Minkowski space-time. Is this unique?

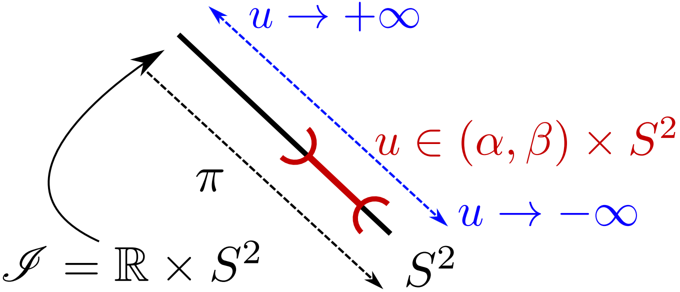
- surely no for any diffeomorphism ϕ will send a solution to another $\phi^*\eta$.

What if we quotient by diffeomorphisms?

- quotienting by all diffeomorphism will give you a unique gravity vacuum
- quotienting only by diffeomorphisms fixing the conformal boundary $\phi\big|_{\mathscr{I}} = Id$ results in the gravity vacua Γ_0!

Memory effect

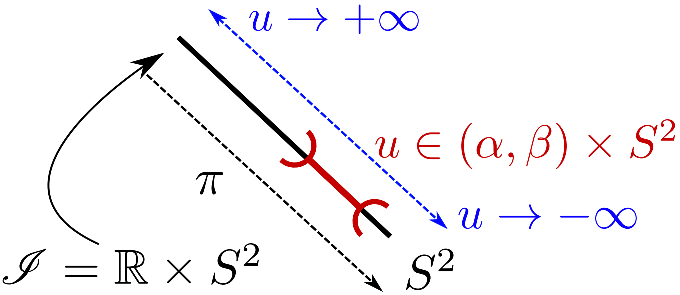
Gravity vacua have the following interesting property: they are completely defined by there value on an open set of the form $(\alpha, \beta) \times S^2$.



i.e if D is flat on $U = (\alpha, \beta) \times S^2$ there is a unique flat extension D_0^U on the whole of $\mathscr{I}$.

Memory effect

Gravity vacua have the following interesting property: they are completely defined by there value on an open set of the form $(\alpha, \beta) \times S^2$.



i.e if D is flat on $U = (\alpha, \beta) \times S^2$ there is a unique flat extension D_0^U on the whole of $\mathscr{I}$.

This is at the origin of a memory effect...

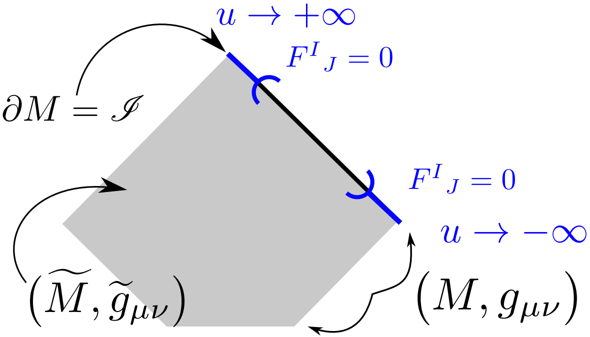
Memory effect

Gravity vacua have the following interesting property: if D is flat on $U = (\alpha, \beta) \times S^2$ there is a unique flat extension D_0^U on the whole of $\mathscr{I}$.

Let D be a null-normal tractor connection corresponding to a "burst" of gravitational waves
i.e such that it is both flat in the "far future" and "far past"
(i.e its curvature is compactly supported on $\mathscr{I}$.)

Memory effect

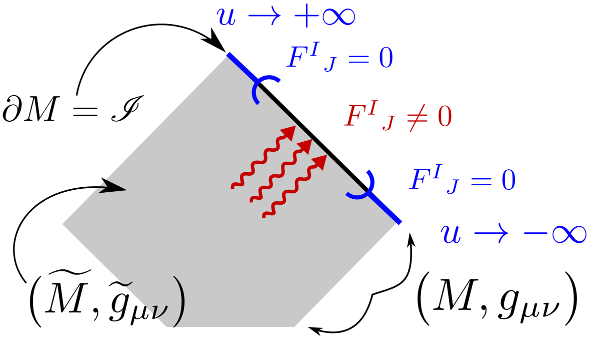
Gravity vacua have the following interesting property: if D is flat on $U = (\alpha, \beta) \times S^2$ there is a unique flat extension D_0^U on the whole of $\mathscr{I}$.



Let D be a null-normal tractor connection corresponding to a "burst" of gravitational waves
i.e such that it is both flat in the "far future" and "far past"
(i.e its curvature is compactly supported on $\mathscr{I}$.)

Memory effect

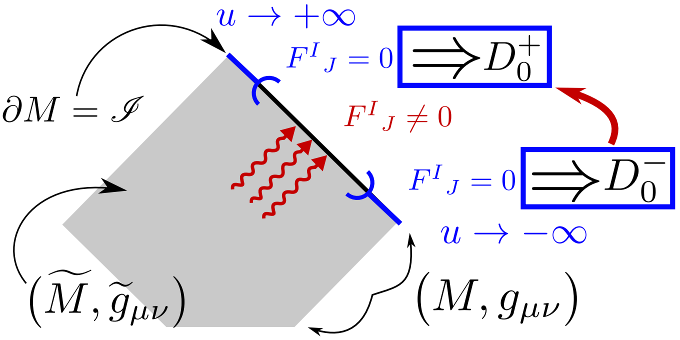
Gravity vacua have the following interesting property: if D is flat on $U=(\alpha,\beta)\times S^2$ there is a unique flat extension D_0^U on the whole of $\mathscr{I}$.



Let D be a null-normal tractor connection corresponding to a “burst” of gravitational waves
i.e such that it is both flat in the “far future” and “far past”
(i.e its curvature is compactly supported on $\mathscr{I}$.)

Memory effect

Gravity vacua have the following interesting property: if D is flat on $U = (\alpha, \beta) \times S^2$ there is a unique flat extension D_0^U on the whole of $\mathscr{I}$.

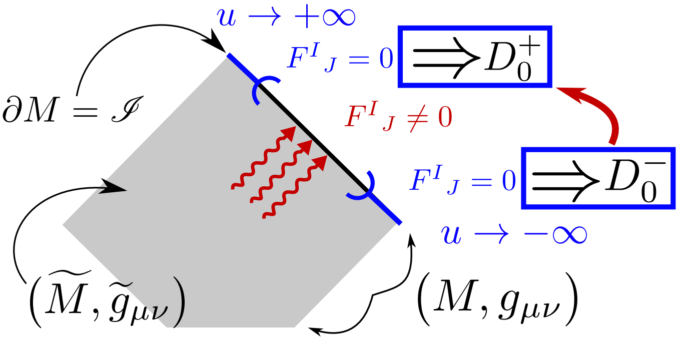


By the above property, such a connection D defines two flat tractor connections $D_0^\pm \in \Gamma_0$ by the requirement that they coincide with D in the far past/future:

$$
\begin{aligned}
D_0^+\big|_{S^2\times(\epsilon,+\infty)} &= D\big|_{S^2\times(\epsilon,+\infty)} \\
D_0^-\big|_{S^2\times(-\infty,\epsilon')} &= D\big|_{S^2\times(-\infty,\epsilon')}
\end{aligned}
$$

Memory effect

Gravity vacua have the following interesting property: if D is flat on $U = (\alpha, \beta) \times S^2$ there is a unique flat extension D_0^U on the whole of $\mathscr{I}$.



Therefore gravitational radiation

has sent one gravity vacua D_0^- to another one D_0^+.

The difference $D_0^+ - D_0^-$ is an invariant of the underlying space-times.

Conclusion and outlook

Conclusion

- The geometry of null-infinity is intrinsically **conformal.**
- I suggest that **tractor calculus** (adapted to degenerate conformal geometries) is best adapted to deal with this difficulty in a **completely invariant** way.
- Gravitational radiation is neatly encoded in the curvature of null-normal tractor connections
- Gravity vacua correspond to the degeneracy of flat tractor connections
- The memory effect is completely transparent in these terms

Outlook

Neat, but what is it good for?

- Probably the only formalism that allows to describe physics at null-infinity completely invariantly
- We[1] have an Einstein-Hilbert variational principle in terms of tractor variables:
  - In principle all physics at null-infinity can thus be reformulated in this way!
  - We[2] are working on computing BMS charges and fluxes.
- Application to holographic duality: The null-normal tractor connection describes the geometrical background to which the boundary theory should be coupled.
- Very versatile formalism: it unifies all cosmological constant and both 3D and 4D space-times. Raise the hope to import ideas from one of these to others.

[1] Upcoming work with C.Scarinci

[2] Upcoming work with R.Ruzziconi

Thank You