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# Gap soliton formation by nonlinear supratransmission in Bragg media

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**Abstract**

A shallow Bragg grating in the nonlinear Kerr regime, submitted to incident continuous wave irradiation at a frequency in a band gap, is shown to switch from total reflection to high transmission for incident energy flux above a threshold. Using the theory of nonlinear supratransmission we prove that: (1) the threshold has an explicit analytic expression in terms of the deviation from the Bragg resonance, (2) the process is not the result of a shift of the gap in the nonlinear dispersion relation, (3) it actually results from a fundamental instability, (4) the transmission does occur by means of gap soliton trains, as experimentally observed [D. Taverner et al., Opt. Lett. 23 (1998) 328], (5) the required energy flux tends to zero close to the band edge. This last property suggests experiments under CW-irradiation at a frequency tuned close to the band edge.


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*Keywords:* Gap soliton; Bragg media; Nonlinear supratransmission

## 1. Introduction

Light propagation in dielectric media with periodically varying index possess extremely interesting properties when Kerr nonlinearities come into play, see, e.g., [1]. In the linear regime it is a *photonic band gap* structure, or Bragg medium, which is totally reflective when irradiated in one of the band gaps. However, when Kerr effect becomes sensible, the Bragg medium may switch from total reflection to *high transmissivity*, as predicted in 1979 [2] and experimentally realized in 1992 [3].

After the pioneering experiments of [3] in steady-state regime, the first experimental observations of nonlinear *pulse propagation* in a fiber Bragg grating have been performed in 1996 [4] under laser pulse irradiation at a frequency near (but not inside) the photonic band gap. In that case the nonlinearity acts as a source of *modulational instability* generating trains of pulses [5]. Particularly interesting is the observation of pulse propagation at a velocity less than the light velocity.

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Later in 1998 a convincing experiment with a fiber Bragg grating demonstrated the repeated formation of light pulses under quasi-continuous wave irradiation *inside* a band gap [6]. This nice experimental evidence of gap soliton formation opens a series of fundamental questions that we intend to solve here, namely: (i) what is the very mechanism at the origin of gap solitons generation, (ii) is it realizable under true CW-irradiation, (iii) can one precisely predict the threshold of incident energy flux above which transmission occurs.

On the basis of the coupled mode approach [7,8] we establish that the mechanism at the origin of the switch from total reflection to transmission is a nonlinear instability. This allows us to derive an explicit expression (formula (8)) for the threshold of incident CW-radiation intensity flux as a function of the departure from the Bragg resonance center. Actually, we shall demonstrate that the rather natural intuition of transparency reached because *nonlinearity shifts the gap* gives a wrong threshold prediction.

Then the soliton produced by nonlinear supratransmission [9,10] are characterized and shown to travel at subluminal velocities (as slow as $0.25c$). Moreover, we demonstrate that the process requires less energy for frequencies close to the band edge, opening the way to experimental realizations. Indeed a CW-radiation, as opposed to a pulsed laser, can be sharply peaked at a given frequency and tuned close to the band edge.

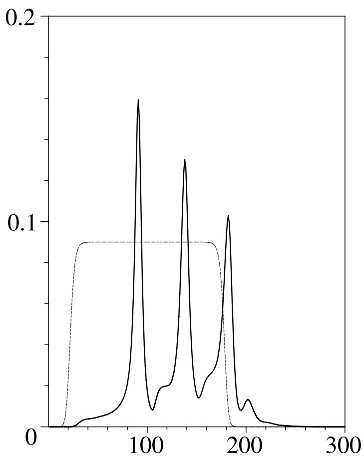


Fig. 1. Energy density output $|e(L,t)|^2$ (full line) compared to the input (dashed line) as functions of normalized time in a typical situation with $\Omega = 0.95$ and $A = 0.30$ in the boundary data (7).

An illustration of the process is displayed on Fig. 1 where the intensity of the output flux, showing a sequence of gap solitons, is depicted together with the CW-input flux (obtained from (9) with parameters $t_m = 100$, $t_0 = 20$, $t_1 = 90$ and $p = 2$ and physical variables $\Omega = 0.95$ and $A = 0.30$). It is instructive to compare (qualitatively) the above plot with the Fig. 3 of Ref. [6], as for their Fig. 5 which compares well with our Fig. 5, showing in particular a transmission factor reaching the value of 40%.

## 2. Nonlinear supratransmission threshold

Our starting point is the basic coupled mode system [7,8] governing the forward $E_f$ and backward $E_b$ slowly varying envelopes of the electric field

$$E(Z,T) = \left[E_f e^{ik_0 Z} + E_b e^{-ik_0 Z}\right] e^{-i\omega_0 T}. \qquad (1)$$

$E(Z,T)$ is the transverse polarized component propagating in the direction $Z$ at frequency close to the Bragg frequency $\omega_0$. Following [8] we write the coupled mode system in reduced units as

$$\begin{aligned} i\left[\frac{\partial e}{\partial t} + \frac{\partial e}{\partial z}\right] + f + \left(\frac{1}{2}|e|^2 + |f|^2\right) e = 0, \\ i\left[\frac{\partial f}{\partial t} - \frac{\partial f}{\partial z}\right] + e + \left(\frac{1}{2}|f|^2 + |e|^2\right) f = 0. \end{aligned} \qquad (2)$$

The reduced units are $z = \kappa Z/c$ and $t = \kappa T$, where $c$ is the light velocity in the medium and $\kappa$ the coupling constant. The reduced field variables are $e = E_f\sqrt{2\Gamma/\kappa}$ and $f = E_b\sqrt{2\Gamma/\kappa}$, where $\Gamma$ is the nonlinear factor.

The (linear) dispersion relation of (2), namely $\omega^2 = 1 + k^2$, possess the gap $[-1, +1]$. In physical units, this gap is $[-\kappa, \kappa]$ and a wave of *frequency* $\Omega$ for system (2) means a field of frequency $\omega_0 + \kappa\Omega$. Note that the characterization of the Bragg medium allows to measure the gap openings which actually furnishes the value of the coupling parameter $\kappa$.

The solitary wave solution of (2) given in [11] has a simple stationary expression

$$e_s(z,t) = \sqrt{\frac{2}{3}}\,\lambda e^{-i(\Omega t - \phi)} \times \cosh^{-1}\left[\lambda(z - z_0) - \frac{i}{2}q\right], \qquad (3)$$

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$$f_s(z,t) = -\sqrt{\frac{2}{3}}\lambda e^{-i(\Omega t-\phi)} \times \cosh^{-1}\left[\lambda(z-z_0)+\frac{i}{2}q\right]. \tag{4}$$

The real-valued parameters determining this 'gap soliton' are $\Omega$ (frequency), $\phi$ (initial phase) and $z_0$ (center), and we have

$$\Omega^2 = 1-\lambda^2, \quad \Omega = \cos q. \tag{5}$$

Then $\lambda$ plays the role of the 'wave number' of the evanescent wave. The "frequency" $\Omega \in [-1,1]$ represents the small deviation $\kappa\Omega$ from the Bragg resonance $\omega_0$.

For a Bragg medium, extending in the region $z \in [0,L]$, and initially *in the dark*, we set the initial data

$$e(z,0) = 0, \qquad f(z,0) = 0. \tag{6}$$

The boundary value problem that mimics the scattering of an incident radiation at frequency $\omega_0 + \kappa c\Omega$ on the medium in $z=0$ is

$$e(0,t) = Ae^{-i\Omega t}, \qquad f(L,t) = 0, \tag{7}$$

where the second requirement means no backward wave incident from the right in $z=L$. The constant $A$ (in general complex valued) is the dimensionless amplitude of the incoming radiation. Eqs. (6) and (7) constitute a well-posed initial-boundary value problem for the PDE (2).

In the linear case such a boundary forcing would simply generate the evanescent wave $e(z,t) = Ae^{-i\Omega t-\lambda z}$. In the nonlinear case, however, this boundary forcing generates the solution (3) for the value of $\phi$ and $z_0 < 0$ such that the amplitude in $z=0$ be precisely $A$. Then, for each fixed forcing frequency $\Omega$, there exists a maximum value $A_s$ of $A$ beyond which there is no solution $\{\phi, z_0\}$. This threshold $A_s$ is given by $|e(0,t)|$ from (3) evaluated at $z_0=0$, namely by

$$A_s = 2\sqrt{2/3}\sin\left(\frac{1}{2}\arccos\Omega\right). \tag{8}$$

This is the threshold amplitude of the incident envelope above which the system develops an instability and generates a propagating nonlinear mode.

Remark: it is a fact that the nonlinearity adapts to the periodic boundary forcing by means of a one-soliton tail and not by means of a multi-soliton solution. Although our numerical simulations provides evidence of this property, its mathematical proof is an open problem of fundamental interest. A qualitative understanding follows from the fact that the one-soliton tail provides an adiabatic deformation of the linear evanescent wave (as amplitude is increased), and that any higher multi-soliton solution would require more energy to be generated.

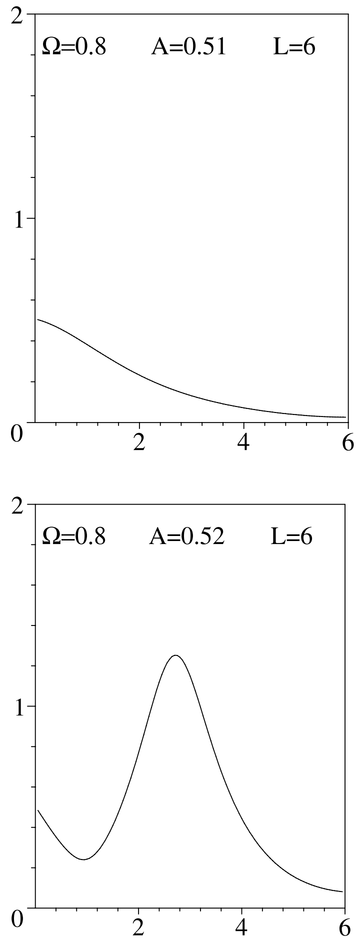


Fig. 2. Plots of the amplitude $|e(z,t)|$ of the right-going envelope at given time ($t=145$) for two values of the driving amplitude $A$ at frequency $\Omega = 0.8$.

Fig. 2 displays the modulus of the right-going envelope for two different numerical simulations of (2) with an incident frequency $\Omega = 0.8$ and with amplitudes $A = 0.51$ (no transmission) and $A = 0.52$ (gap soliton generation) when the theoretical threshold predicted by (8) is $A_s = 0.5164$.

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In order to avoid initial shock, the boundary condition $e(0,t)$ is turned on and off *smoothly* by assuming instead of (7)

$$e(0,t) = \frac{A}{2}\big[\tanh\big(p(t-t_0)\big) - \tanh\big(p(t-t_1)\big)\big] \times e^{-i\Omega t}. \tag{9}$$

A practical interest of such an incident wave is that it reproduces the *quasi-continuous wave* irradiation of the experiments of [6]. Most of the results presented here are obtained with $N = 120$ spatial mesh points, $h = 0.05$ grid spacing over (hence a length $L = 6$ normalize units) a time of integration $t_m = 200$. The parameters of the boundary field are a real-valued amplitude $A$, a slope $p = 0.2$, an ignition time $t_0 = 20$ and an extinction time $t_1 = 180$.

The above simulations provide a simple diagnostic to test the response of the medium to at a given CW-radiation of frequency $\omega_0 + \kappa c\Omega$ and of varying intensities. This will be done in the next section after having found the threshold amplitude that would result from the intuitive requirement of *nonlinear shift* of the gap.

## 3. Nonlinear dispersion relation

The prediction of transparency of a Bragg mirror by nonlinear shift effect in the dispersion relation is obtained by seeking a solution

$$e = a\exp\big[i(kz-\omega t)\big], \qquad f = b\exp\big[i(kz-\omega t)\big]$$

of (2). It furnishes the following nonlinear algebraic system for the unknowns $a$ (incident amplitude) and $\omega$ (incident frequency)

$$(\omega - k)a + b + \left(\frac{1}{2}a^2 + b^2\right)a = 0,$$
$$(\omega + k)b + a + \left(\frac{1}{2}a^2 + b^2\right)b = 0. \tag{10}$$

This system must be interpreted as giving the *non-linear dispersion relation* $\omega(a,k)$ expressed in terms of the wave number $k$ once the amplitude $b$ has been eliminated.

This can be done by simple algebraic manipulations and the resulting relation is a third-order algebraic equation for $\omega(k,a)$ with 3 real-valued solutions. One of those 3 solutions must be discarded as it is singular in the linear limit $a \to 0$, the others two giving the deformations of the linear branches.

However, as we are interested essentially in the shift of the gap, it is sufficient to study the solutions at $k = 0$ where the system simplifies to (assuming naturally $ab \neq 0$)

$$2\omega = -\left(\frac{b}{a} + \frac{a}{b}\right) - \frac{3}{2}(a^2 + b^2), \tag{11}$$
$$\left(1 + \frac{ab}{2}\right)(b^2 - a^2) = 0. \tag{12}$$

The second equation has the 3 solutions $b = \pm a$ and $b = -2/a$. As stated before, this last solution diverges when $a$ vanishes, it is unphysical.

The two physical solutions $b = \pm a$ correspond to the upper branch ($\omega > 0$ i.e. $b = -a$) and the lower one ($\omega < 0$ i.e. $b = a$). As we consider right-going incident waves, we stick with $b = -a$ for which (12) readily provides the value of the gap opening

$$\omega(0,a) = 1 - \frac{3}{2}a^2. \tag{13}$$

This simply states that the bottom of the passing band is lowered from the (linear) value 1 by the nonlinear shift $3a^2/2$.

For our purpose it is more convenient to express the value $A_m$ of the incident amplitude $a$ for which the incident frequency $\Omega$ falls in the allowed band namely

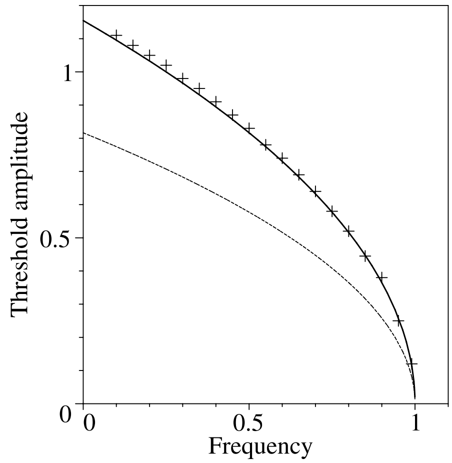


Fig. 3. Values of the threshold of nonlinear supratransmission observed on numerical simulations (crosses) as compared with expression (8) (solid line) and with (14) (dashed line).

$\Omega = \omega(0, a)$. This provides

$$A_m = \sqrt{\frac{2}{3}(1 - \Omega)} \tag{14}$$

as the prediction of *transparency* by nonlinear shift of the gap.

We are able now to compare our prediction (8) of amplitude threshold both to the above value (14) and to results of numerical simulations of the system (2). This is done for a series of frequency values in the range $[0.1, 0.995]$ and we obtain the Fig. 3. The dots represent the smallest value (with absolute precision of $10^{-2}$) of the amplitude $A$ for which nonlinear supratransmission is seen to occur. The expression (14) is also plotted (dashed line) which shows that the *nonlinear shift* provides a wrong prediction.

## 4. Generating instability

The underlying fundamental mechanism of soliton generation is a nonlinear instability [12] which we work out by proving that the first order correction to the nonlinear evanescent wave $(e_s, f_s)$ given in (3), (4) experiences exponential growth as soon as its position $z_0$ becomes positive. Inserting

$$e = e_s + \epsilon\phi_1 e^{-i\Omega t}, \qquad f = f_s + \epsilon\phi_2 e^{-i\Omega t}, \tag{15}$$

in (2) one gets at first order system in $\epsilon$. This system can be written for the 2-vector $\phi(z,t) = {}^{\mathrm{T}}(\phi_1, \phi_2)$ as

$$z > 0: \quad i(\sigma_3\partial_t + \partial_z)\phi + K\phi + U(z)\phi = V(z)\bar{\phi}, \tag{16}$$

where $\sigma_3$ is the Pauli matrix and where

$$K = \begin{pmatrix} \Omega & 1 \\ -1 & -\Omega \end{pmatrix}, \qquad U = \begin{pmatrix} 2|u|^2 & -u^2 \\ \bar{u}^2 & -2|u|^2 \end{pmatrix},$$
$$V = \begin{pmatrix} -\frac{1}{2}u^2 & |u|^2 \\ -|u|^2 & \frac{1}{2}\bar{u}^2 \end{pmatrix},$$
$$u(z) = \lambda\sqrt{\frac{2}{3}}\cosh^{-1}\left[\lambda(z - z_0) - \frac{i}{2}q\right]. \tag{17}$$

The free parameters are $\Omega$ and $z_0$ and we have $q = \arcsin(\Omega)$ and $\lambda = \sin(q)$.

Eq. (16) maps to a spectral problem on the semi-line $z > 0$ by seeking eigenfunctions

$$\phi(z,t) = \varphi_1(z)e^{i\omega t} + \varphi_2(z)e^{-i\omega t}. \tag{18}$$

The resulting eigenvalue problem can then be written for the 4-vector $\Psi = {}^{\mathrm{T}}(\varphi_1, \varphi_2)$ as

$$z > 0: \quad i\partial_z\Psi + [\mathcal{K} + \mathcal{U}(z)]\Psi = 0,$$
$$\mathcal{K} = \begin{pmatrix} K - \sigma_3\omega & 0 \\ 0 & K + \sigma_3\omega \end{pmatrix},$$
$$\mathcal{U}(z) = \begin{pmatrix} U & -V \\ -V & U \end{pmatrix}. \tag{19}$$

The *spectral parameter* $k$ is obtained by looking at the large $z$ asymptotic behavior. As $\mathcal{U}(z) \to 0$ for $z \to \infty$, we write $\Psi \sim e^{-ikz}$ at large $z$ and the complex scalar $k$ is then defined by

$$\det\{\mathcal{K} + k\} = 0 \iff (\Omega \pm \omega)^2 = 1 + k^2. \tag{20}$$

The localized potential $\mathcal{U}(z)$ possess bound states in the upper half complex $k$-plane whose locations vary with the free parameter $z_0$ (for each given frequency $\Omega$). We argue that when $z_0$ becomes positive, the related eigenvalue makes the eigenfrequency $\omega$ to become imaginary hence generating an exponential growth of the perturbation (15). The analytical proof requires the explicit solution of (19) which then allows to define the Jost function whose poles in the upper half $k$-plane are the bound states locations. This can be performed only in a few simple cases, as done in

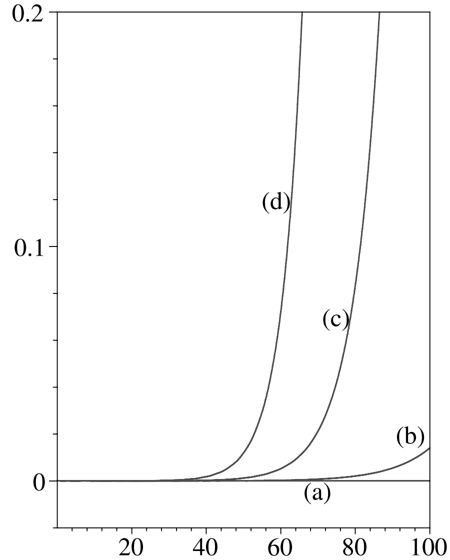


Fig. 4. Plot of $|\phi(1,t)|^2$ solution of (16) in the cases (a) $z_0 = 0$, (b) $z_0 = 0.1$, (c) $z_0 = 0.15$, (d) $z_0 = 0.2$, when $\Omega = 0.5$ and $L = 6$ for an initial condition $\phi(z, 0) = 0.01$.

[12] and the above $4 \times 4$ eigenvalue problem is an open problem. However, numerical simulations of the linear system (16) do show the onset of exponential growth of the solution $\phi(z,t)$ for a generic initial-boundary condition as soon as $z_0 > 0$. We have plot on Fig. 4 the value of $|\phi|^2$ at one point $z = 1$ as a function of $t$ obtained by solving the linear PDE (16) for initial value $\phi(z,0) = 0.01$ and different values of the position $z_0$ as indicated.

## 5. Bifurcation of transmitted energy

The system (2) possess the conservation law

$$\partial_t\big(|e|^2 + |f|^2\big) + \partial_z\big(|e|^2 - |f|^2\big) = 0. \tag{21}$$

For a given boundary condition (7) (i.e., for fixed $A$ and $\Omega$), we define the incident ($I$), reflected ($R$) and transmitted ($T$) energies at given arbitrary time $t_m$ by

$$I(A,\Omega) = \int_0^{t_m} dt\,\big|e(0,t)\big|^2, \tag{22}$$

$$R(A,\Omega) = \int_0^{t_m} dt\,\big|f(0,t)\big|^2, \tag{23}$$

$$T(A,\Omega) = \int_0^{t_m} dt\,\big|e(L,t)\big|^2. \tag{24}$$

Then, upon integration of (21) on the length $[0, L]$ and time $[0, t_m]$ we obtain

$$R + T - I + \int_0^{L} dz\,\Big(\big|e(z,t_m)\big|^2 + \big|f(z,t_m)\big|^2\Big) = 0. \tag{25}$$

If $t_m$ is larger than the irradiation duration, the energy injected eventually radiates out completely, namely $e(z,t_m)$ and $f(z,t_m)$ vanish, and we are left with the conservation relation $R + T = I$ which can be written

$$\rho(A,\Omega) + \tau(A,\Omega) = 1, \quad \rho = \frac{R}{I}, \quad \tau = \frac{T}{I}. \tag{26}$$

Fig. 5 shows a typical numerical simulation where, for $\Omega = 0.95$, we have computed the reflection and transmission factors, together with their sum, for 50 different values of the amplitude $A$ in the range $[0.20, 0.45]$. These simulations show the sudden energy flow through the medium, as a result of nonlinear supratransmission. Note that the conservation (26) hold rigorously only for $t_m \to \infty$ which is not the case in numerical simulations and which explains the small variations seen on Fig. 5 of the sum $\rho + \tau$.

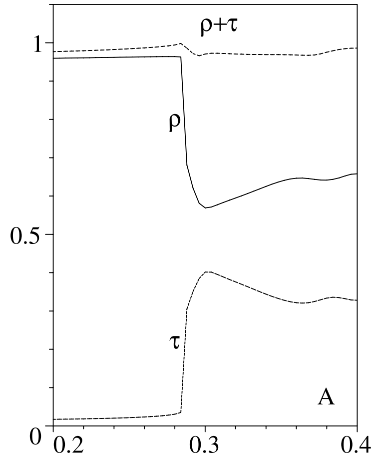

Fig. 5. Plot of the reflection factor $\rho$ (full line), transmission $\tau$ (dashed line) and sum $\rho + \tau$ (dotted line) for system (2) with boundary value (9) and parameter values given there.

The transmissivity is due to the generation and propagation of light pulses shown in Fig. 1 representing the energy density $|e(L,t)|^2$ measured at the output as a function of time. We show now that these are *gap solitons* travelling at a fraction of the light velocity (slow light pulses).

## 6. Travelling gap solitons

Although the system (2) is not Lorentz invariant, the propagating solution of [11] can still be written in terms of the boosted variables [13]. As we are interested here only in the energy flux, we write the soliton solution $|e(z,t)|^2$ moving at velocity $v < 1$ with frequency $\cos q$ as

$$\big|e(z,t)\big|^2 = \frac{2\big(\frac{1-v^2}{3-v^2}\big)\big(\frac{1+v}{1-v}\big)^{1/2}\sin^2 q}{\big|\cosh\big(\frac{\sin q}{\sqrt{1-v^2}}(z - z_0 - vt) - \frac{i}{2}q\big)\big|^2}. \tag{27}$$

Such an expression allows to fit a given simulation by seeking the two parameters $v$ and $q$, and the initial

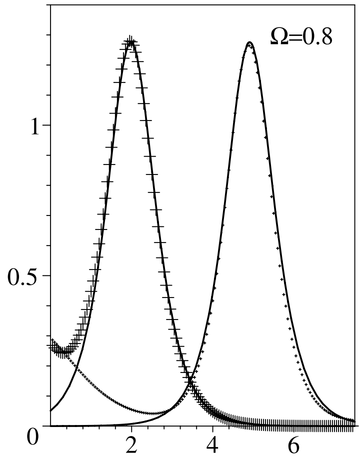


Fig. 6. Fit of the numerical solution (crosses) by analytical soliton solution (solid line) at times 65.7 and 72.7, for an incident frequency $\Omega = 0.8$ and amplitude $A = 0.53$.

position $z_0$, that reproduce with the explicit solution the results of the numerical simulation.

An instance of such a fit is displayed in Fig. 6 that shows the analytic soliton $|e(z,t)|^2$ compared with the numerical simulation for two fixed values of time. This furnishes the following velocities:

| | | | | | | |
|---|---|---|---|---|---|---|
| $\Omega$ | 0.95 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 |
| $A$ | 0.28 | 0.38 | 0.53 | 0.65 | 0.75 | 0.83 |
| $v$ | 0.25 | 0.34 | 0.42 | 0.50 | 0.58 | 0.66 |

(28)

These are *slow light pulses* and naturally, small velocities are obtained close to the gap edge where the required energy is small.

## 7. Conclusion

We have demonstrated that a Bragg medium in the nonlinear Kerr regime irradiated by a CW-beam becomes a *soliton generator* by means of nonlinear supratransmission for an incident intensity flux above an explicit threshold (not predicted by nonlinear dispersion relation). It provides, in particular, the first analytic expression of the minimum energy flux required to observe soliton generation from an incident CW-irradiation in terms of the departure from the Bragg gap center.

While experimental search for gap soliton was mainly performed by means of pulsed laser, our result indicates that using a CW laser beam is the good choice as it can be finely tuned to a frequency that would require low energy flux. This constitutes a practical tool to investigate switching properties of a Bragg medium and allows at the same time to understand the process at the origin of sudden transmissivity.

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