Large amplitude surface waves in infinite depth, a variational approach

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A variational scheme describing 2-dimensional surface waves on deep water is devised. It uses as a fundamental ansatz the functional form of the linear wave pattern extended to the large amplitude domain. Surprisingly, this scheme turns out to be entirely solvable by means of elementary analytical methods, and enables us to reproduce very closely the aspect ratio H/L corresponding to the peaking of the wave, long known to be 14% (see reference [4]). It also incorporates (at a very naive level) some aspects of the breaking of waves.

PACS numbers: 47.35.+i, 47.10.+g,45.10.Db

I. INTRODUCTION

Variational methods are often powerful tools to compute approximate results, in cases where no small parameter is available to apply some form of perturbation theory. It is therefore tempting to extend the use of these methods to various domains in Physics. This can certainly be done for theories that can be expressed in terms of a lagrangian variational principle : instead of extremizing over the whole set of functions (in an appropriate functional space), whose extremum is the exact (but unknown) solution satisfying the Euler-Lagrange equation, one restricts the search of the extremum to a small class of function. This restricted class is often expressed in terms of an ansatz function depending on a few free parameters, and the search is performed in a finite dimensional space, where the optimization procedure is tractable. The advantages of the method lie in its simplicity and flexibility : a great variety of ansätze can be devised for a given physical problem, in order to highlight a peculiar aspect of the solution, such as its general form, the place of its singularities etc...

In the present article we consider the problem of stationary wave pattern propagating in deep water. The Lagrangian formulation of the inviscid, irrotational two dimensional fluid flow has a long and well established story (see e.g. [2] and references therein). We shall apply the variational method in a non trivial situation, by considering the large amplitude case where the exact flow is not known in a closed form. However, a great number of results, concerning in particular the form of the maximal amplitude wave have been obtained in the past, starting from the pioneering works of Stokes [3] and Michell [4], up to recent results on the form of the crest at the peak [5]-[7]. Moreover the form and the velocity of the profile of the wave has been calculated using numerical methods (see for example reference [9]). All these results will enable us to check the validity of our approximate variational solution.

In the following, we will use the general functional form of the small amplitude solution $\phi \sim e^{\kappa z}\, u(x-ct)$, where ϕ is the stream function of the velocity field, as our ansatz (This type of approach has been already been used in different context [1]). The variational parameters will consist of the unknown function u, together with the inverse damping height κ . Surprisingly enough, the variational problem turns out to be entirely solvable by means of elementary analytical methods, for every values of the wave velocity c and a large range of wave amplitude H. We will therefore be able to check the a priori obvious fact that our method reproduces the linear small amplitude wave pattern exactly, as it should. We will also compare this variational wave pattern with the reference results mentioned above, and ascertain its validity with an accuracy of the order 5% in the large amplitude limit, when the peaking of the wave is attained.

II. FUNDAMENTAL VARIATIONAL ANSATZ

We study two dimensional stationary waves propagating with celerity c along the x direction. The fluid extends from $z=-\infty$ to the surface $z=h(x-ct)$. Let us denote the components of the velocity field by

$$\vec{v} = v(x-ct,z)\,\mathbf{e_x} \,+\, w(x-ct,z)\,\mathbf{e_z}\ . \tag{II.1}$$

In order to implement a consistent variational scheme in the deep water waves problem, we start from the Lagrangian density for the 2-dimensional Euler equations

$$\mathcal{L} = \int_{-\infty}^{h} (v^2+w^2)\,dz \;-\; gh^2 \;+\; \mu\,(h_t + v\,h_x - w)_{z=h}\ , \tag{II.2}$$

where, in order to incorporate from the beginning the incompressibility hypothesis, we express the two components of the fluid velocity in term of a stream function $\phi(x,z,t)$:

$$v \;=\; -\phi_z \quad , \quad w \;=\; \phi_x\ . \tag{II.3}$$

In this Lagrangian, the first (bulk) term provides the irrotationality equation $\Delta\phi = 0$, the second one accounts for the dynamic free surface condition, and the last constraint term represents the kinematic free surface condition. The mean water level is given by the constant solution $h(x,t)=0$.

We now choose as variational ansatz a stream function

$$\phi(x,z,t) = -\frac{1}{\kappa}\, e^{\kappa\, z}\; u(x-ct) \tag{II.4}$$

which reproduces the fundamental ansatz of [1],

$$\begin{cases} v \;=\; -\phi_z \;=\; e^{\kappa\, z}\, u(x-ct) \\ w \;=\; \phi_x \;=\; -\frac{1}{\kappa}\, e^{\kappa\, z}\, u_x(x-ct)\ . \end{cases} \tag{II.5}$$

In this expression c is the a priori chosen velocity of the stationary wave pattern, while the inverse damping height κ and the velocity profile function u will be determined by the variational condition. This form of ansatz will guaranty the incompressibility condition $v_x+w_z=0$. It incorporates the z-dependence of the linear wave pattern, which we expect to be reasonably correct also for non-linear waves.

III. AN EXPLICIT VARIATIONAL SOLUTION

We now proceed to the solution of the previous variational scheme, which turns out to be simple enough to present an explicit analytical solution.

In a first step, the propagation hypothesis $h_t=-c\,h_x$ enables us to write the kinematic constraint equation as an exact x-derivative, that we can solve in the form

$$\kappa\,(h-h_0)\,e^{-\kappa\, h} = \frac{u}{c}\ . \tag{III.1}$$

The constant of integration h_0 will be fixed later by imposing the mean level condition

$$\int h\, dx = 0\ . \tag{III.2}$$

We now use equation (III.1) to express the Lagrangian (II.2) as a function of the water height $h(x-ct)$ only

$$\mathcal{L} \;=\; \frac{c^2}{2\,\kappa}\,\Big(\kappa^2\,(h-h_0)^2 \;+\; h_x^2\,(1-\kappa(h-h_0))^2 - \frac{2\,\kappa\, g}{c^2}\, h^2\Big)\ . \tag{III.3}$$

Let us make the following change of variables

$$\kappa x=\xi\ ,\quad ct=\tau\ ,\quad \eta(\xi,\tau)=\kappa h(x,t)\ ,\quad \eta_0=\kappa h_0 \tag{III.4}$$

$$\psi(\xi,\tau)=\eta(\xi,\tau)-\eta_0 \tag{III.5}$$

$$\alpha=(\frac{2}{r}-1)\ ,\quad \lambda=\frac{\eta_0}{(r/2-1)} \tag{III.6}$$

with the dimensionless constant $r=\kappa c^2/g$ (note that in the linear case, we expect $r=1$ and $\alpha=1$). We can now write $\mathcal{L}$ in the simple form

$$\mathcal{L} \;=\; \frac{c^2}{2\,\kappa}\,\Big\{\psi_\xi^2(1-\psi)^2-\alpha(\psi-\lambda)^2-\lambda^2(\frac{r}{2}-1)\Big\}\ . \tag{III.7}$$

In this last equation, we have kept the "constant" term depending on κ. Although this term is inessential in determining the wave profile ψ, it is crucial for the determination of the optimal value of κ.

The variational equations obtained from $\mathcal{L}$ are totally similar to those of a one dimensional problem of classical mechanics, where the ξ coordinate plays the role of the time, and the water height ψ corresponds to the dynamical variable. This can be solved with the help of the "energy conservation" equation

$$\psi_\xi^2+\frac{\alpha(\psi-\lambda)^2-2A}{(1-\psi)^2}=0\ , \tag{III.8}$$

the "energy" constant A being related to the amplitude of the wave. This equation can be viewed as the zero-energy solution of a non-linear oscillator in a potential $V(\psi)=(\alpha(\psi-\lambda)^2-2A)/(1-\psi)^2$ shown in figure 1.

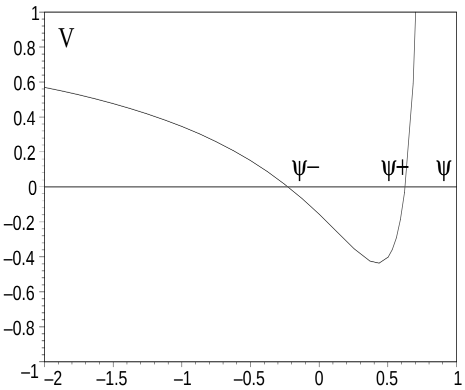

FIG. 1: The potential corresponding to the oscillator described by equation (III.8).

Let us call $\psi_\pm=\lambda\pm\sqrt{2A/\alpha}$ the two turning points of the movement (see figure 1).The total amplitude of the wave follows immediately

$$H=(\psi_+-\psi_-)/\kappa=2\,\frac{\sqrt{2A/\alpha}}{\kappa}\ . \tag{III.9}$$

Moreover, the wave profile can be obtained from equation (III.8) by direct integration : choosing the ξ origin at the crest of the wave, we have

$$\xi=\int_\psi^{\psi_+}\frac{d\psi}{\sqrt{V(\psi)}}\ , \tag{III.10}$$

which finally yields under a simple change of variable

$$\begin{cases} \psi \;=\; \lambda+\sqrt{2A/\alpha}\,\chi \\ \sqrt{\alpha}\,\xi=(1-\lambda)\arccos(\chi)-\sqrt{2A/\alpha}\,\sqrt{1-\chi^2}\ . \end{cases} \tag{III.11}$$

If we let $\chi = \cos(t)$ we immediately recognize in this wave profile a cycloid, that can be written, using the physical dimensions h and x and the total amplitude H as

$$\begin{cases} h &= H/2\,\cos(t) \\ x &= L/(2\pi)\,t - H/(2\sqrt{\alpha})\,\sin(t) \end{cases} , \qquad \text{(III.12)}$$

where the length of the wave L is given by the arch length of the cycloid :

$$L = \frac{2\pi(1-\lambda)}{\sqrt{\alpha}\kappa} . \qquad \text{(III.13)}$$

The mean water level is then given by

$$<\psi> = \frac{\int \psi d\xi}{\int d\xi} = \lambda - \frac{A}{\alpha(1-\lambda)} , \qquad \text{(III.14)}$$

which, with the help of equations (III.4)-(III.6), and the level condition (III.2), enables us to fix the constant λ (or equivalently the constant h_0 appearing in the equation (III.1)) :

$$\lambda(1-\lambda) = \frac{2A}{\alpha r} = \frac{\kappa H^2 g}{4c^2} . \qquad \text{(III.15)}$$

Retaining the solution that goes continuously to $\lambda = 0$ in the limit of linear waves $H = 0$ (see paragraph below), we finally obtain λ in term of physical quantities as

$$\lambda = (1 - \sqrt{1 - \frac{gH^2\kappa}{c^2}})/2 . \qquad \text{(III.16)}$$

Up to now, the variational procedure has been performed with respect to the function u which appears in the equation (II.4), with the remarkable result of a complete analytical form for the wave profile (see equation (III.11)). To determine the inverse damping height κ, we should in principle insert the wave profile (III.11) in the Lagrangian (II.2), integrate over a wave length L, and extremize this quantity with respect to κ. Although this could be done (actually, we have done it as a check), it is a much simpler task to use another equivalent variational quantity with respect to which we determine κ, namely the wave length L itself. Using the equations (III.13) and (III.16), we have

$$L = \pi\,\frac{1 + \sqrt{1 - \frac{gH^2\kappa}{c^2}}}{\sqrt{\frac{2\kappa g}{c^2} - \kappa^2}} . \qquad \text{(III.17)}$$

In terms of the dimensionless quantities $r = \kappa c^2/g$ and $a = gH/c^2$, the optimum equation $\partial L/\partial\kappa = 0$ reads

$$\frac{a^2 r^2}{2} + (1-r)(1 + \sqrt{1 - a^2 r}) = 0 . \qquad \text{(III.18)}$$

Solving for r in this last equation furnishes the optimal value of κ which in turn leads to a unique optimal wave profile, through (III.11)-(III.1)-(II.5) and the definitions (III.4)-(III.6). This has been done in figure 2, for a wave velocity $c = 10\,m.s^{-1}$, and a wave height $H = 7.1\,m$. After the completion of the variational procedure, we find a cycloid profile, with a wave length $L = 53.8\,m$, corresponding to an aspect-ratio $\rho = H/L = 13\%$ very close to the peak limit 14.2% of reference [4]. It is also useful to compare this result to the value that we get in the linear case (see paragraph below) $L_{lin} = 64.0\,m$. Finally we find for the inverse penetration depth the value $\kappa = 0.12\,m^{-1}$, the linear result being $k_{lin} = 0.098\,m^{-1}$.

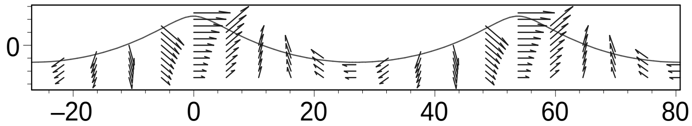

FIG. 2: The optimal wave profile corresponding to a wave velocity $c = 10\,m.s^{-1}$ and a wave height $H = 7.1\,m$. The corresponding wave length is $L = 53.8\,m$, while the inverse penetration depth is $\kappa = 0.12\,m^{-1}$. (note that the x and z scales are different)

IV. THE LINEAR LIMIT

To check the validity of the procedure, it is useful to examine the linear limit, corresponding to small amplitudes, namely for a wave height H very small compared to the (linear) wave length. Let us call

$$k_{lin} \equiv g/c^2 \qquad \text{(IV.1)}$$

the wave vector in the linear limit. We have then $H << 1/k_{lin}$, or $a << 1$. From this, equation (III.18) trivially gives $r = 1$, or equivalently $\kappa = k_{lin}$. Reporting this result in equation (III.17), we get for the optimal wave length

$$L = \frac{2\pi}{k_{lin}} = \frac{2\pi c^2}{g} . \qquad \text{(IV.2)}$$

Lastly, using in this case the values $\alpha = 1$ and $\lambda = 0$, obtained from equations (III.6) and (III.16), equation (III.11) reproduces the linear wave profile

$$h(x - ct) = \frac{H}{2}\cos(k_{lin}(x - ct)) \qquad \text{(IV.3)}$$

as it should. Of course, the complete velocity profile obtained by using equation (II.5), corresponding by construction to the linear case, completes the picture .

V. THE NONLINEAR DISPERSION RELATION

We may now ask the natural question : how does the non-linearity modify the linear dispersion relations (IV.1)-(IV.2)?

A first answer is given by figure 3, where we have plotted the value of the optimal wavelength L given by equations (III.17) and (III.18), as a function of the wave height H, for a fixed value of the wave celerity $c = 10\, m.s^{-1}$. For comparison, we show also the constant value of L corresponding to the small amplitude case (equation (IV.2)). This result, together with the aspect of the wave profile displayed in figure 2, clearly shows the important modification brought in by the finite amplitude H [10].

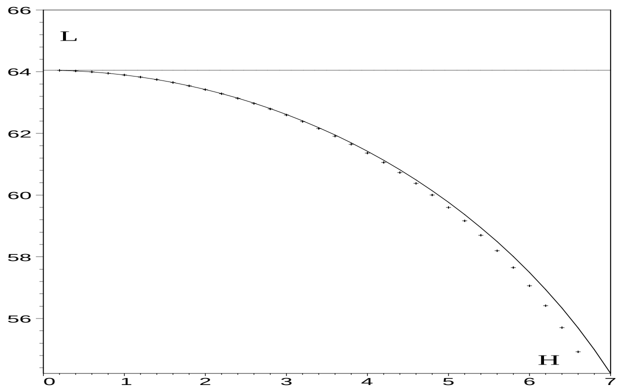

FIG. 3: The optimal wave length L versus the wave height H for a velocity $c = 10\, m.s^{-1}$. The horizontal straight line corresponds to the linear limit. The doted line corresponds to the relation $L = 2\pi/\kappa_{opt}$.

The optimal inverse damping length κ_{opt}, given by the solution of the equation (III.18) is represented by the doted line of figure 3, where the value $2\pi/\kappa_{opt}$ has been plotted versus H. This last curve shows that the linear relation between the inverse penetration case and the wave length $L\kappa_{opt} = 2\pi$ holds almost unchanged in the non linear case, although these two quantities differ substantially from their linear values.

Finally, from equation (III.18) the wave velocity as a function of the wave amplitude H is given by the simple expression

$$c^2 = \frac{g}{\kappa} \, \frac{1}{1-(\kappa H/2)^2} \, , \tag{V.1}$$

provided that $\kappa H \leq \sqrt{2}$.

This relation is displayed in figure 4, where, in order to compare this to the reference results found in the standard textbook [9], we have plotted the quantity $c/c_{lin} - 1$ versus $kH/2$. Note that this last curve reproduces closely the curve displayed in [9]-p.453, and obtained by "a very intricate evaluation of Fourier coefficients and the use of ingenious methods of high accuracy for accelerated summation of series." Compared to the simplicity of our computation, it demonstrates the effectiveness of the variational approach.

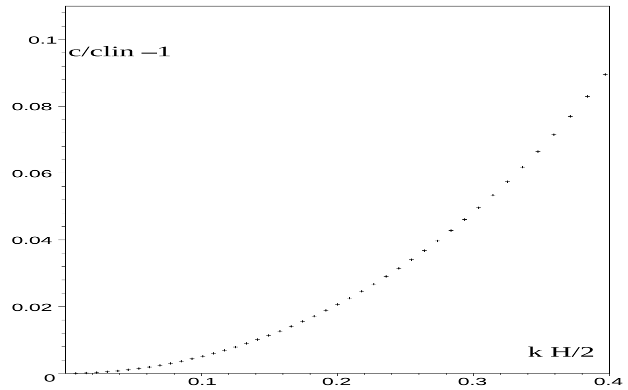

FIG. 4: The wave velocity as a function of the wave amplitude H.

VI. PEAKING AND BREAKING

For large values of the wave height H, the optimal wave pattern develops a singularity for the wave, or peaking. Then the wave reaches its maximal height, and the corresponding aspect ratio $\rho = H/L$ has been calculated long ago to be $\rho = 0.142$ [4]. Using the cycloidal wave profile of equation (III.11), it is easy to see that the peak is attained when

$$(1-\lambda) = \sqrt{\frac{2A}{\alpha}} = \frac{\kappa H}{2} = \frac{ar}{2} \, . \tag{VI.1}$$

Using equation (III.15) this condition can be rewritten in term of the adimensional variables $r \sim \kappa$ and $a \sim H$ as

$$r = \frac{2}{a} - 1 \tag{VI.2}$$

which corresponds to the singular profile plotted in figure 5-(a).

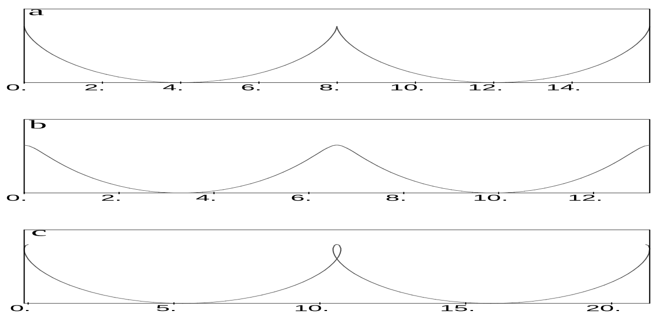

FIG. 5: The different optimal profiles corresponding to a) the singular profile corresponding to the peaking of the wave, b) a regular wave profile, c) a non-physical profile leading to breaking. (x and z scales are different).

We immediately note that the form of the wave at the peak is not the one that is expected since, according to the work of Stokes, the angle at the crest should be 120^o

(see reference [8]). Actually, we find the form of the singularity to be $h(x) \sim |x|^{2/3}$, leading to a to much more pronounced peak, but this problem only affects a small region of the profile, the global form leading to a correct aspect ratio, as we will see below.

Lower values of r than the one given by equation (VI.2) lead to a regular hypo-cycloidal profile, shown in figure 5-(b), while greater ones produce an hyper-cycloidal wave pattern, a non physical feature that we interpret as the sign of the breaking of the waves, and that is displayed in figure 5-(c).

Up to now, nothing has been said about the possibility for these different profiles to correspond to an extremum, namely to fulfill the condition (V.1). In order to see which form of wave does effectively leads to an extremum of the action, we rewrite this condition, using $0 \leq f \equiv \kappa H \leq \sqrt{2}$ as

$$f\,(1 - f^2/4) = a\ . \tag{VI.3}$$

From this last elementary form, and the peaking condition (VI.2), the following results follow :

- for $a < 1/\sqrt{2} \simeq 0.707$, equation (VI.3) has a single positive root $f_- < \sqrt{2}$, corresponding to an unique extremal profile of the regular type shown in figure 5-(b).
- for $1/\sqrt{2} \leq a < 3 - \sqrt{5} \simeq 0.764$ there are two positive roots $0 < f_- < f_+ < \sqrt{2}$, leading to two extrema of the action. f_+ is related to the "breaking" situation of figure 5-(c), while f_- is always regular. Actually the profiles displayed in figures 5-(b)-(c) are the ones associated to the same value of the wave height $a = 0.74$.
- for $a = 3 - \sqrt{5}$ the unstable profile f_+ give birth to the peaking situation, shown in figure 5-(a). We can then compute the aspect ratio for this extremum:

$$\rho = \frac{H}{L} = \frac{1}{\pi}\sqrt{\frac{7 - 3\sqrt{5}}{\sqrt{5} - 1}} \simeq 0.155 \tag{VI.4}$$

which compares well to the reference value 0.142 of [4]. We can also compute the ratio of the wave velocity c to the value $c_{lin} = \sqrt{g/\kappa}$ corresponding to the linear case. We find at the peak the value

$$\frac{c}{c_{lin}} = \sqrt{\frac{2\pi\rho}{a}} = 1.128\ . \tag{VI.5}$$

"Extensive numerical computations" provide the value $c/c_{lin} = 1.092$ (see e.g. reference [9]), which differs from our result by only 4%. These two results enable us to make contact with a realistic physical situation, and to validate our approach with a precision of the order of 5%.

- for $3 - \sqrt{5} \leq a < \frac{4}{3\sqrt{3}} \simeq 0.770$ the two profiles are regular, and finally coalesce into a single one for $a = \frac{4}{3\sqrt{3}}$. For greater amplitudes, there is no more extremum of the action.

We now present these results in figure 6, which shows the different extremal wave lengths L as a function of the wave amplitude H, for a wave velocity of $c = 10\,m.s^{-1}$.

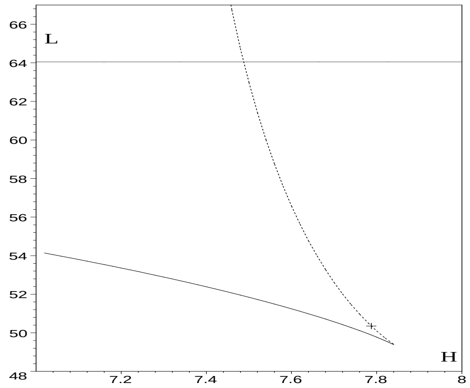

FIG. 6: The optimal wave length L versus the wave height H for a velocity $c = 10\,m.s^{-1}$. The horizontal straight line corresponds to the linear limit, the solid line to the regular optimum already shown in figure 3, the dotted one to the other extremum. The cross on this branch corresponds to the singular profile (peaking).

In this graph, the horizontal line gives the linear result. The solid line corresponds to the "regular" extremum given by f_-, already shown in figure 3. The form of this profile is always hypo-cycloidal, as in figures 2 and 5-(b). The dotted line corresponds to the "non-physical" hyper-cycloidal profile, which appears for $H > 1/\sqrt{2}c^2/g \simeq 7.21\,m$ (figure 5-(c)). This profile eventually gives birth to a singular peaking form (figure 5-(a)), signaled here by a cross, for $H = (3 - \sqrt{5})c^2/g \simeq 7.79\,m$. It then evolves into a normal hypo-cycloidal profile before merging with the first regular one for $H = 4/3/\sqrt{3}c^2/g \simeq 7.85\,m$. No extremum exists for larger H.

We note that the singular profile does not appear at the point where the two extrema coalesce. However we interpret the closeness between this point and the extremity of the two branches, and the correct value of the aspect ratio obtained there, as the sign that our variational approach is effective.

Finally, it is interesting to study the angle of the wedge of the wave for large amplitude. Actually, the value $\theta = 120^0$ obtained by Stokes at the peak corresponds to an unstable profile, as shown in [6]. However solutions asymptotically close to the Stokes profile but with a rounded peak have been shown to exist [5], and our optimal profiles bear some ressemblance to these solutions, as we now show : from the wave profile III.12, the position of the inflexion point is given by the condition (see figure 7)

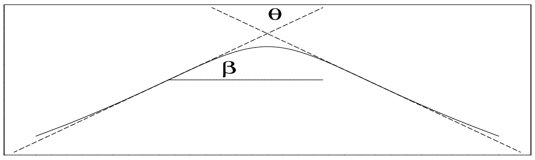

FIG. 7: The form of the cycloidal wave profile near the wedge (un-scaled coordinates).

$$\frac{d^2h}{dx^2} = 0 \,, \tag{VI.6}$$

which implies

$$\cos(t) = \frac{\pi H}{L\sqrt{\alpha}} \,, \tag{VI.7}$$

and the slope of the wave at this point is

$$\tan(\beta) = \frac{\pi H}{\sqrt{L^2 - \pi^2 H^2/\alpha}} \,. \tag{VI.8}$$

In figure 8 we display the wedge angle $\theta = \pi - 2\beta$ for the two branches of extrema of figure 6. for $H \simeq 0$ the profile is sinusoidal and flat, and $\theta \simeq 180^0$. This angle goes down as H is increased, until it reaches the value $\theta = 0$ at the peak.

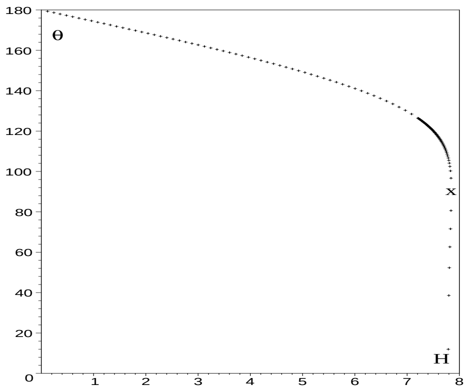

FIG. 8: The angle θ of the wedge of the wave as a function of the wave height H. The cross corresponds to the maximal wave amplitude in our variational scheme.

Although the Stokes result $\theta = 120^0$ is not obtained at the peak, as previously stated, we find that the change of slope which happens in figure 8 precisely near that value is highly suggestive. This value is obtained on the regular branch, very near from the apex in figure 6, represented here by a cross. This is a further indication that, although the fine details of the wave pattern could not be obtained within the framework of such a crude approach, the global features of the flow are well reproduced by our method.

VII. CONCLUSION

An important issue to this work is the possibility to extend such a variational approach to different physical situations. In particular, it would be interesting to look at the finite depth case, without restriction to shallow water. Our approach is totally suitable in that situation. Indeed, although the computations appear in a less pleasing form, and involve some numerical computation, we have been able to obtain the same kind of results as in this work, going continuously from the deep water case to the shallow water one. This will be the subject of a next publication. Of particular interest is the possibility to develop solitons in that case, and the prediction of the critical depth at which such a phenomenon occurs.

Another issue concerns the precise form of the wave crest when the peak phenomenon occurs, particularly in view of the wealth of results obtained in that case (see e.g. [5]-[7]). As we have stated before, the approach developed in this paper works well for the global wave pattern, but fails to reproduce the detailed form of the peak correctly. However we hope to report soon some results developed with an ansatz tailored to study that precise situation.

Acknoledgements : I am grateful to A. Neveu for prompting my interest to this subject and for fruitful discussions, and to J. Leon for many valuable suggestions.

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