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ZAKHAROV–SHABAT SPECTRAL TRANSFORM ON THE HALF-LINE

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The Zakharov–Shabat inverse spectral problem is constructed for a potential with support on the half-line and with a boundary value at the origin. This prescribed value is shown to produce a Jost solution with an essential singularity at large values of the spectral parameter; this requires particular attention when solving the related Hilbert boundary value problem. The method is then used to illustrate the sine-Gordon equation (in the light cone) and is discussed using a singular limit of the stimulated Raman scattering equations.

Keywords: nonlinear evolution equations, inverse scattering transform, boundary value problem, Riemann–Hilbert problem, sine-Gordon equation

1. Introduction

Extending the integrability of a nonlinear evolution equation from the infinite line to a finite line (or the half-line at least) is an old problem that has recently attracted much attention. For instance, the *stimulated Raman scattering* (SRS) system, which has long been known to have a Lax pair [1], was shown to be also solvable on the half-line only recently [2] (also see [3]). In this case, the boundary data induce formation of solitons from the initial vacuum, which can have interesting physical implications.

Another widely studied integrable evolution equation is the nonlinear Schrödinger equation (NLS) [4], [5], whose solvability by the *inverse spectral transform* (IST) on the infinite line is understood as a Dirichlet condition on the U-shaped domain $\{x \in \mathbb{R}, t = 0\} \cup \{x = \pm\infty, t > 0\}$. The IST then allows constructing the solution $q(x,t)$ at any t for all x that vanishes with all its derivatives as $x \to \infty$ if it vanishes at the initial time. The Lax pair therefore furnishes a time evolution of the spectral transform that preserves the properties essential for reconstructing the solution of the NLS (by the inverse problem) in a given class of functions.

But on the half-line, because of the nature of the Lax pair, the time evolution of the spectral transform requires not only the the boundary value at $x = 0$ but also the value of the derivative. One of the main problems to solve is to express the derivative somehow in terms of the boundary data; this problem has motivated a number of interesting works. The Dirichlet boundary problem was studied in detail in [6], where it was shown that the problem of determining the derivative is transformed into the problem of determining some *missing spectral data*. A different method, with similar results but based on the GLM technique, was developed in [7] in the case of the Korteweg–de Vries equation.

An interesting approach was proposed in [8], where both operators of the Lax pair were treated as joint spectral problems. Applied to the NLS, this method provides the solution as a system of coupled integral equations. Although difficult to use, the method allows a comprehensive spectral characterization and gives particular solutions of the NLS in some cases. This method has also proved very useful in treating linear

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partial differential equations with moving boundaries [9] and was successfully applied to the sine-Gordon equation in the light cone, for which the Lax pair is compatible with the boundary values [8], [10].

In all these works, the essential tool is the Zakharov–Shabat spectral problem on the half-line $x > 0$ where the *potential* $q(x,t)$ has a given boundary value $q_0(t)$ at $x = 0^+$. It recently appeared that the related Hilbert boundary value problem that links the Jost solutions to the spectral transform must be solved in a particular way to take the singular behaviors of these Jost solutions at large values of the spectral parameter into account. The solution was given in [11] and is used with the same notation throughout the present work.

The purpose of this paper is to briefly present the solution of the Hilbert boundary value problem related to the Zakharov–Shabat system on the half-line and apply it to the sine-Gordon equation in the light cone. The time evolution of the spectral transform is then shown to generate an infinite number of discrete eigenvalues in the complex plane, as already demonstrated in [6], which raises the question of whether the IST method is adequate. In particular, for a constant boundary value, the infinite set of poles is created *instantaneously*. This question is then considered using the SRS that yields the sine-Gordon equation in some limit. We discover that this limiting process regularizes the spectral transform, even in the case of a constant boundary value for which the poles are created in pairs at successive different times.

2. The spectral transform

For completeness, we devote this section to recalling the essentials of the theory of the spectral transform for the Zakharov–Shabat operator on the half-line [11] and to setting the definitions and notation.

2.1. Basic solutions. The time t, being a free parameter, is omitted throughout this section. We consider the Zakharov–Shabat spectral problem [4] on the half-line

$$\phi_x = ik[\phi, \sigma_3] + Q(x)\phi, \qquad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \tag{2.1}$$

with the given boundary values

$$q(x)\big|_{x=0^+} = q_0, \qquad r(x)\big|_{x=0^+} = r_0. \tag{2.2}$$

The Jost solutions [12] can be defined as the solutions of the integral equations

$$\begin{pmatrix} \psi_{11}^+(k,x) \\ \psi_{21}^+(k,x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \displaystyle\int_0^x dy\, q(y)\psi_{21}^+(k,y) \\ \displaystyle\int_0^x dy\, r(y)\psi_{11}^+(k,y)e^{2ik(x-y)} \end{pmatrix}, \tag{2.3}$$

$$\begin{pmatrix} \phi_{12}^+(k,x) \\ \phi_{22}^+(k,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \displaystyle -\int_x^\infty dy\, q(y)\phi_{22}^+(k,y)e^{-2ik(x-y)} \\ \displaystyle\int_0^x dy\, r(y)\phi_{12}^+(k,y) \end{pmatrix} \tag{2.4}$$

$$\begin{pmatrix} \phi_{11}^-(k,x) \\ \phi_{21}^-(k,x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \displaystyle\int_0^x dy\, q(y)\phi_{21}^-(k,y) \\ \displaystyle -\int_x^\infty dy\, r(y)\phi_{11}^-(k,y)e^{2ik(x-y)} \end{pmatrix}, \tag{2.5}$$

$$\begin{pmatrix} \psi_{12}^-(k,x) \\ \psi_{22}^-(k,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \displaystyle\int_0^x dy\, q(y)\psi_{22}^-(k,y)e^{-2ik(x-y)} \\ \displaystyle\int_0^x dy\, r(y)\psi_{12}^-(k,y) \end{pmatrix}. \tag{2.6}$$

The so-called physical solution

$$\Phi(k,x) = (\phi_1^-, \phi_2^+) \tag{2.7}$$

satisfies the conditions

$$\Phi \underset{x=0}{\longrightarrow} \begin{pmatrix} 1 & \rho^+ \\ \rho^- & 1 \end{pmatrix} \underset{x\to\infty}{\longrightarrow} \begin{pmatrix} \tau^- & 0 \\ 0 & \tau^+ \end{pmatrix}, \tag{2.8}$$

where the *reflection coefficients* $\rho^\pm(k)$ are defined from the above integral equations at $x = 0^+$, namely,

$$\rho^+ = -\int_0^\infty dx\, q\phi_{22}^+ e^{2ikx}, \qquad \rho^- = -\int_0^\infty dx\, r\phi_{11}^- e^{-2ikx}, \tag{2.9}$$

and the *transmission coefficients* $\tau^\pm(k)$ by taking the limit as $x \to \infty$, i.e.,

$$\tau^+ = 1 + \int_0^\infty dx\, r\phi_{12}^+, \qquad \tau^- = 1 + \int_0^\infty dx\, q\phi_{21}^-. \tag{2.10}$$

By direct computation with (2.1), we demonstrate that the determinant of any solution is independent of x and can therefore be calculated at $x = 0$ or at $x \to \infty$. Applied to Φ, this gives

$$1 - \rho^+\rho^- = \tau^+\tau^-, \tag{2.11}$$

which is called the *unitarity relation*. The rest of the solution, namely,

$$\Psi(k,x) = (\psi_1^+, \psi_2^-), \tag{2.12}$$

is used to solve the inverse problem, and we call it the *intermediate solution*. For $k \in \mathbb{R}$, it satisfies the conditions

$$\Psi \underset{x=0}{\longrightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underset{x\to\infty}{\longrightarrow} \begin{pmatrix} \dfrac{1}{\tau^+} & \dfrac{-e^{-2ikx}\rho^+}{\tau^+} \\ \dfrac{-e^{2ikx}\rho^-}{\tau^-} & \dfrac{1}{\tau^-} \end{pmatrix}. \tag{2.13}$$

The solutions Φ and Ψ are linearly dependent, and by matching their values at $x = 0^+$ (or, equivalently, at $x = \infty$), we readily obtain

$$\Phi = \Psi e^{-ik\sigma_3 x} \begin{pmatrix} 1 & \rho^+ \\ \rho^- & 1 \end{pmatrix} e^{ik\sigma_3 x}. \tag{2.14}$$

The above expression (defining the *transfer matrix*) can also be expressed as the *Hilbert boundary value problem* [13] on the real axis

$$\begin{aligned} \psi_1^+ - \phi_1^- &= -e^{2ikx}\rho^-\psi_2^-, \\ \phi_2^+ - \psi_2^- &= e^{-2ikx}\rho^+\psi_1^+. \end{aligned} \tag{2.15}$$

To solve the above system, we need the properties of its solutions as functions of the complex parameter k.

2.2. Analytic properties. Using the standard method [12], we find that for potentials vanishing sufficiently rapidly as $x \to \infty$, the vectors ψ_1^+ and ψ_2^- are analytic in the respective upper and lower halves of the k plane. Next, the vector ϕ_2^+ is a meromorphic function of k in the upper half-plane with a finite number N^+ of simple poles k_n^+ that are the zeros of the analytic function $1/\tau^+(k)$. Indeed, it follows from (2.13) that

$$\frac{1}{\tau^+} = 1 + \int_0^\infty q\psi_{21}^+ \tag{2.16}$$

with ψ_{21}^+ analytic. Moreover, the vector $\tilde{\phi}_2^+ = \phi_2^+/\tau^+$ can be written from (2.4) as the solution of

$$\begin{pmatrix} \tilde{\phi}_{12}^+(k,x) \\ \tilde{\phi}_{22}^+(k,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \displaystyle\int_0^x dy\, q(y)\tilde{\phi}_{22}^+(k,y)e^{-2ik(x-y)} \\ \displaystyle\int_0^x dy\, r(y)\tilde{\phi}_{12}^+(k,y) \end{pmatrix}, \tag{2.17}$$

which shows that $\tilde{\phi}_2^+$ is analytic in k and that ϕ_2^+ therefore has poles where the analytic function $1/\tau^+(k)$ has zeros.

Similarly, ϕ_1^- is meromorphic in the lower half-plane with N^- poles k_n^-, which are the zeros of $1/\tau^-(k)$. Consequently, from their definitions (2.9) and (2.10), $\tau^\pm$ and $\rho^\pm$ have meromorphic extensions to their respective half-planes $\pm\,\mathrm{Im}(k) > 0$, where they have the $N^\pm$ simple poles $k_n^\pm$ (the locations of bound states). In particular, we have the relation obtained by simply taking the residues of (2.14),

$$\operatorname*{Res}_{k_n^+}\{\phi_2^+\} = \rho_n^+\psi_1^+(k_n^+)e^{-2ik_n^+x}, \qquad \rho_n^+ = \operatorname*{Res}_{k_n^+}\{\rho^+\}, \tag{2.18}$$

$$\operatorname*{Res}_{k_n^-}\{\phi_1^-\} = \rho_n^-\psi_2^-(k_n^-)e^{2ik_n^-x}, \qquad \rho_n^- = \operatorname*{Res}_{k_n^-}\{\rho^-\}. \tag{2.19}$$

The spectral data is then given by the set

$$\mathcal{S} = \{\mathrm{Im}(k) > 0,\ \rho^+(k); \mathrm{Im}(k) < 0,\ \rho^-(k)\}, \tag{2.20}$$

which is complete if we prove that it allows reconstructing the potentials $r(x)$ and $q(x)$ with $q(0^+) = q_0$ and $r(0^+) = r_0$.

2.3. Large *k* behaviors. To solve Hilbert problem (2.15) with the singularities $k_n^\pm$ in the complex plane, we need the behavior of the solutions as $|k| \to \infty$. Integrated by parts, integral equations (2.4) and (2.6) provide the behaviors (in $\mathrm{Im}(k) \le 0$ for the vectors with the minus superscript and in $\mathrm{Im}(k) \ge 0$ for the plus superscript):

$$(\phi_1^-, \phi_2^+) = \mathbf{1} + \frac{1}{2ik}\begin{pmatrix} -\displaystyle\int_0^x rq & q \\ -r & \displaystyle\int_0^x rq \end{pmatrix} + \mathcal{O}\left(\frac{1}{k^2}\right), \tag{2.21}$$

$$(\psi_1^+, \psi_2^-) = \mathbf{1} + \frac{1}{2ik}\begin{pmatrix} -\displaystyle\int_0^x rq & q - q_0e^{-2ikx} \\ -r + r_0e^{2ikx} & \displaystyle\int_0^x rq \end{pmatrix} + \dots. \tag{2.22}$$

Consequently, when q_0 and r_0 do not vanish, the intermediate solution has an essential sigularity on the real axis, which must be reproduced by the solution of the Hilbert problem as demonstrated in [11]. We also have

$$\begin{aligned} \rho^+(k) &= \frac{1}{2ik}q_0 - \frac{1}{(2ik)^2}q_0' + \mathcal{O}\left(\frac{1}{k^3}\right), \\ \rho^-(k) &= -\frac{1}{2ik}r_0 - \frac{1}{(2ik)^2}r_0' + \mathcal{O}\left(\frac{1}{k^3}\right), \end{aligned} \tag{2.23}$$

where we define $q_0' = q_x(0^+, t)$.

2.4. Solution of the Hilbert problem. We consider the set of *spectral data*

$$\mathcal{S} = \{\mathrm{Im}(k) \geq 0,\ \rho^+(k);\ \mathrm{Im}(k) \leq 0,\ \rho^-(k)\}, \tag{2.24}$$

where $\rho^+(k)$ and $\rho^-(k)$ are continuous and bounded on the real axis and meromorphic in the respective upper and lower half-planes with finite numbers of single poles k_n^+ and k_n^- and related residues ρ_n^+ and ρ_n^-. Moreover, in their respective half-planes, they have behaviors (2.23), where q_0 and r_0 are given numbers.

The problem is to construct the two matrices $\Phi(k,x)$ and $\Psi(k,x)$, solutions of Hilbert problem (2.15) having (*i*) the prescribed analytic properties, (*ii*) the boundary behaviors for large k given in (2.21) and (2.22), and (*iii*) boundary values (2.8) and (2.13). This problem was solved in [11], and the solution is given by

$$\phi_1^-(k,x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi}\int_{-\infty+i0}^{+\infty+i0} \frac{d\lambda}{\lambda - k}\rho^-(\lambda)e^{2i\lambda x}\psi_2^-(\lambda,x) - \\ - \sum_{n=1}^{N^-} \frac{\rho_n^-}{k_n^- - k}\psi_2^-(k_n^-,x)e^{2ik_n^- x}, \quad \mathrm{Im}(k) \leq 0, \tag{2.25}$$

$$\phi_2^+(k,x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi}\int_{-\infty-i0}^{+\infty-i0} \frac{d\lambda}{\lambda - k}\rho^+(\lambda)e^{-2i\lambda x}\psi_1^+(\lambda,x) - \\ - \sum_{n=1}^{N^+} \frac{\rho_n^+}{k_n^+ - k}\psi_1^+(k_n^+,x)e^{-2ik_n^+ x}, \quad \mathrm{Im}(k) \geq 0, \tag{2.26}$$

where the function $\Psi(k,x)$ solves the system of Cauchy–Green integral equations

$$\psi_1^+(k,x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2i\pi}\int_{-\infty-i0}^{+\infty-i0} \frac{d\lambda}{\lambda - k}\rho^-(\lambda)e^{2i\lambda x}\psi_2^-(\lambda,x) - \\ - \sum_{n=1}^{N^-} \frac{\rho_n^-}{k_n^- - k}\psi_2^-(k_n^-,x)e^{2ik_n^- x}, \quad \mathrm{Im}(k) \geq 0, \tag{2.27}$$

$$\psi_2^-(k,x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2i\pi}\int_{-\infty+i0}^{+\infty+i0} \frac{d\lambda}{\lambda - k}\rho^+(\lambda)e^{-2i\lambda x}\psi_1^+(\lambda,x) - \\ - \sum_{n=1}^{N^+} \frac{\rho_n^+}{k_n^+ - k}\psi_1^+(k_n^+,x)e^{-2ik_n^+ x}, \quad \mathrm{Im}(k) \leq 0. \tag{2.28}$$

In contrast to what occurs in the one-dimensional case $x \in (-\infty,+\infty)$, the integrals here are taken over not the real axis but contours along the real axis in the upper or lower half-plane as indicated.

Substituting behavior (2.21) in (2.1), we then obtain the potential from the relation

$$Q(x) = i[\sigma_3, \Phi^{(1)}(x)], \tag{2.29}$$

where we recall that $\Phi^{(1)}(x)$ denotes the coefficient of $1/k$ in the large-k expansion of the solution $\Phi(k,x)$. In other words, we have

$$q(x) = -\frac{1}{\pi}\int_{-\infty-i0}^{+\infty-i0} \rho^+(\lambda)\psi_{11}^+(\lambda,x)e^{-2i\lambda x} + 2i\sum_{n=1}^{N^+}\rho_n^+\psi_{11}^+(k_n^+,x)e^{-2ik_n^+ x} \tag{2.30}$$

for $q(x)$, which can also be written as (ρ has no poles on the real axis)

$$q(x) = -\frac{1}{\pi} \int_{C_+} \rho^+(\lambda)\psi_{11}^+(\lambda, x)e^{-2i\lambda x}, \tag{2.31}$$

where C_+ is a smooth contour in the upper half-plane extending from $-\infty + i0$ to $+\infty + i0$ and passing *above* all the poles k_n^+. The above expression can be understood as the inverse of (2.9), namely, of

$$\rho^+(k) = -\int_0^\infty dx\, q(x)\phi_{22}^+(k, x)e^{2ikx}. \tag{2.32}$$

It is then clear that as soon as a t dependence is assumed for q, there is a corresponding t dependence of the spectral transform ρ. The central point of the IST method consists in expressing this link (which is almost trivial in the infinite-line case).

2.5. Reductions. The particular evolution that we study results from a *reduction* from the two potentials r and q to only one and goes in two steps. First,

$$\overline{Q} = \sigma_2 Q \sigma_2 \quad \Leftrightarrow \quad r = -\bar{q}, \tag{2.33}$$

which implies that

$$\Psi(k, x) = \sigma_2 \overline{\Psi(\bar{k}, x)} \sigma_2. \tag{2.34}$$

Examination of their boundary behavior as $x \to \infty$ gives the reduction constraint on the spectral transform

$$\overline{\tau^-(\bar{k})} = \tau^+(k), \qquad \overline{\rho^-(\bar{k})} = -\rho^+(k). \tag{2.35}$$

Considering the poles and residues, we then have

$$N^+ = N^-, \qquad \overline{\rho_n^-} = -\rho_n^+, \qquad \overline{k_n^-} = k_n^+. \tag{2.36}$$

In what follows, we write ρ for ρ^+ and k_n for k_n^+.

The second reduction is given by

$$Q = \sigma_2 Q \sigma_2 \quad \Leftrightarrow \quad r = -q. \tag{2.37}$$

In this case, it is easy to prove that

$$\Psi(k, x) = \sigma_2 \Psi(-k, x) \sigma_2, \tag{2.38}$$

and examination of the boundary behavior as $x \to \infty$ and $x \to 0$ gives the reduction constraint on the spectral transform

$$\begin{aligned} &\rho^+(k,t) = -\rho^-(-k,t), \qquad \tau^-(-k) = \tau^+(k), \\ &N^+ = N^-, \qquad \rho_n^- = -\rho_n^+. \end{aligned} \tag{2.39}$$

In the case of the sine-Gordon equation, the above two reductions in Eqs. (2.33) and (2.37) hold together, which actually implies that $q \in \mathbb{R}$. As a consequence, by direct manipulations with relations (2.35) and (2.39), the resulting constraint on the spectral transform $\rho = \rho^+$, which arises in addition to (2.39), is given by

$$\bar{\rho}(-\bar{k}) = \rho(k). \tag{2.40}$$

Consequently, the poles in the complex plane come in pairs: if k_n is a pole of $\rho(k)$ with the residue ρ_n, then $-\bar{k}_n$ is also a pole with the residue $-\bar{\rho}_n$. In short,

$$\operatorname*{Res}_{k_n}\{\rho(k)\} = \rho_n \quad \Rightarrow \quad \operatorname*{Res}_{-\bar{k}_n}\{\rho(k)\} = -\bar{\rho}_n. \tag{2.41}$$

3. The Dirichlet problem for the sine-Gordon equation

3.1. Boundary values and the Lax pair. The above tools can be successfully applied to the sine-Gordon equation in the light cone

$$\theta_{xt} + \sin\theta = 0, \tag{3.1}$$

which is here solved for the Dirichlet conditions $\theta_0(t)$ and $\tilde{\theta}(x)$, namely,

$$t \in [0,T]\colon \theta(0,t) = \theta_0(t), \qquad x \in [0,\infty)\colon \theta(x,0) = \tilde{\theta}(x), \tag{3.2}$$

in the class of functions vanishing (modulo 2π) with all derivatives as $x \to \infty$ for every value of time t (the value of T is arbitrary).

The Lax pair found in [14] is given by the Zakharov–Shabat equation for a particular choice of the potentials r and q, namely,

$$\Phi_x = ik[\Phi, \sigma_3] + Q\Phi, \qquad Q = -\frac{i}{2}\sigma_2\theta_x, \tag{3.3}$$

together with the evolution

$$\Phi_t = -\frac{i}{4k}(\sigma_3\cos\theta + \sigma_1\sin\theta)\Phi + \Phi e^{-ik\sigma_3 x}\Omega e^{ik\sigma_3 x}. \tag{3.4}$$

We have chosen to write this Lax pair for the particular solution Φ defined in the previous section.

3.2. Evolution of the spectral transform. Using behaviors (2.8) in (3.4), we calculate Ω and obtain the time evolution of the spectral transform. Within reduction relations (2.39), we eventually obtain

$$\Omega = -\frac{i}{4k}\begin{pmatrix} \cos\theta_0 - \rho(-k,t)\sin\theta_0 & 0 \\ 0 & -\cos\theta_0 + \rho(k,t)\sin\theta_0 \end{pmatrix}, \tag{3.5}$$

$$\rho_t = \frac{i}{4k}\left[\rho^2\sin\theta_0 - 2\rho\cos\theta_0 - \sin\theta_0\right]. \tag{3.6}$$

We note that the reduction constraint $\bar{\rho}(-\bar{k}) = \rho(k)$, which ensures real values of the potential $\theta(x,t)$, is preserved by the above time evolution (as long as $\theta_0 \in \mathbb{R}$, of course). At the singular point $k = 0$, the spectral transform has the regular behavior [11]

$$\rho(0,t) = \frac{\cos\theta_0 - 1}{\sin\theta_0}. \tag{3.7}$$

3.3. An explicit example. To illustrate the above method, we consider the explicitly solvable case of piecewise constant boundary data $\theta_0(t)$, namely,

$$\begin{aligned} t \in [0, t_1]\colon &\qquad \theta_0(t) = 0, \\ t \in [t_j, t_{j+1}]\colon &\qquad \theta_0(t) = \varphi_j, \\ t \in [t_n, \infty]\colon &\qquad \theta_0(t) = 0, \end{aligned} \tag{3.8}$$

compatible with the vanishing initial condition

$$\theta(x,0) = \tilde{\theta}(x) = 0. \tag{3.9}$$

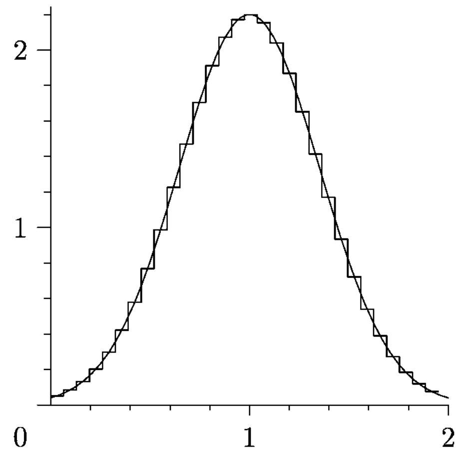

Fig. 1

We note that this boundary data represents an arbitrary localized function $\theta_0(t)$ discretized with an n-step time grid (discretization of the initial impulse having a Gauss form is given in Fig. 1).

We adopt the notation

$$\begin{aligned} &\rho_j(t) = \rho(k,t) \quad \text{for} \quad t \in [t_j, t_{j+1}], \\ &\rho_j = \rho_j(t_{j+1}) \equiv \rho(k, t_{j+1}). \end{aligned} \tag{3.10}$$

For $t \in [t_j, t_{j+1}]$, the solution of evolution (3.6) can then be written as

$$\rho_j(t) = -\frac{b_j(\rho_{j-1}+a_j)e^{i(t-t_j)/2k} - a_j(\rho_{j-1}+b_j)}{(\rho_{j-1}+a_j)e^{i(t-t_j)/2k} - (\rho_{j-1}+b_j)}, \tag{3.11}$$

where

$$a_j = -\frac{1+\cos\varphi_j}{\sin\varphi_j}, \qquad b_j = \frac{1-\cos\varphi_j}{\sin\varphi_j}. \tag{3.12}$$

We note that in the limit as $\varphi_j \to 0$, Eq. (3.11) becomes the solution of evolution (3.6) where $\theta_0 = \varphi_j$ is set to zero.

The large-k behavior of solution (3.11) is obtained by induction and is given by

$$\rho_j(t) = -\frac{i}{4k}\left[(t-t_j)\sin\varphi_j + \sum_{\ell=1}^{j}(t_\ell - t_{\ell-1})\sin\varphi_{\ell-1}\right] + \mathcal{O}\left(\frac{1}{k^2}\right). \tag{3.13}$$

Hence, we have

$$\partial_t\rho_j(t) = -\frac{i}{4k}\sin\varphi_j + \mathcal{O}\left(\frac{1}{k^2}\right), \tag{3.14}$$

which implies that (3.11) has the expected large-k behavior.

At the last step, i.e., for $\varphi_n = 0$, Eq. (3.11) can be written as

$$\rho_n(t) = -\frac{(b_n\rho_{n-1}-1)e^{i(t-t_n)/2k} - a_n\rho_{n-1}+1}{(\rho_{n-1}+a_n)e^{i(t-t_n)/2k} - (\rho_{n-1}+b_n)}, \tag{3.15}$$

which eventually simplifies to

$$\rho_n(t) = \rho_{n-1}e^{i(t-t_n)/2k}. \tag{3.16}$$

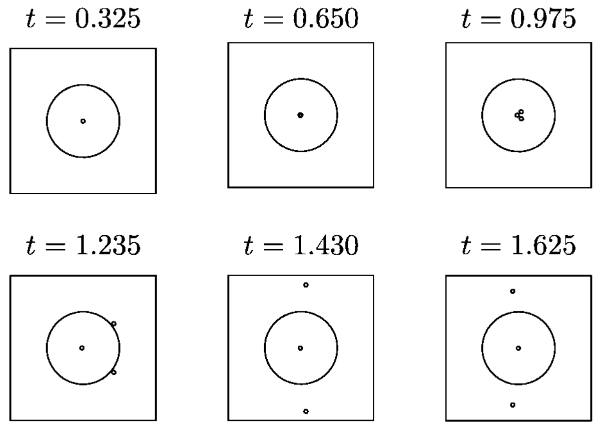

Fig. 2

As a consequence, the spectral transform evaluated *after* the total effect of the boundary data $\theta_0(t)$, i.e., for $t > t_n$, is completely determined by $\rho_{n-1} = \rho(k, t_n)$. This function of k is itself obtained as the *mapping*

$$\rho_0 = 0,$$
$$\rho_j = -\frac{b_j(\rho_{j-1} + a_j)e^{i(t_{j+1}-t_j)/2k} - a_j(\rho_{j-1} + b_j)}{(\rho_{j-1} + a_j)e^{i(t_{j+1}-t_j)/2k} - (\rho_{j-1} + b_j)}, \quad j = 1, \ldots, n-1. \tag{3.17}$$

This is an explicit expression that allows obtaining an insight into the effect of generic input data $\theta_0(t)$ on the vanishing background under a suitable (arbitrary) discretization. We relegate the study of the above mapping (in the complex k plane) to forthcoming work and here simply mention that the main point concerns the study of the creation (annihilation) of poles of $\rho(k,t)$ in the upper half of the k plane.

Recalling reality constraint (2.41), we see that the poles of $\rho(k)$ come in pairs, and a pole $k = k_n(t)$ of $\rho_j(t)$ is given by

$$e^{i(t-t_j)/(2k_n)} = \frac{\rho_{j-1} + b_j}{\rho_{j-1} + a_j}, \tag{3.18}$$

where ρ_{j-1} is itself a function of k_n. For any (smooth and localized) boundary data with a sufficiently large amplitude, there are an infinite number of such poles in the upper half-plane.

Figure 2 shows an instance of the case where the time step $t_{j+1} - t_j = \delta$ is constant for all j. The poles are shown in the plane $\zeta = e^{i\delta/(2k)}$ and belong to the upper k plane when outside the unit circle. In the inverse transformation from ζ to k, it is clear that one pole in the ζ plane corresponds to a countably infinite set in the k plane (with the point $k = 0$ as an accumulation point). Then the theory apparently fails because the Jost eigenfunction is not continuous in a neighborhood of the real axis. We now wish to consider this question in the light of the SRS system and its sine-Gordon limit.

4. Dirichlet problem for the SRS

4.1. Boundary values and the Lax pair. The formalism is now applied to the SRS system for the envelopes of the pump pulse $a(k,x,t)$, the Stokes pulse $b(k,x,t)$, and the medium excitation $q(x,t)$,

$$\partial_x a = q b e^{2ikx}, \qquad \partial_x b = -\overline{q} a e^{-2ikx},$$
$$\partial_t q = -\int g(k)\, dk\, a\overline{b} e^{-2ikx}, \tag{4.1}$$

where $g(k)$ measures the coupling (related to Raman gain) and the parameter k is a mismatch wave number resulting from the group-velocity dispersion [2]. In [1], the Lax pair for the SRS was written in the case where $g(k) = \delta(k - k_0)$, and it was regularized in [15] via the parameter k, whose physical meaning was found in [2].

The Dirichlet conditions are prescribed on the line $x = 0$, $t > 0$,

$$a(k,x,t)|_{x=0} = A(t), \qquad b(k,x,t)|_{x=0} = B(t), \tag{4.2}$$

and on the line $x > 0$, $t = 0$,

$$q(x,t)|_{t=0} = Q(x). \tag{4.3}$$

The given initial value $Q(x)$ is assumed to be of the Schwartz type (it decreases at large x with all its derivatives faster that any polynomial).

The time dependence is inserted in the spectral data by requiring that the solution $\mu^+ = (\psi_1^+, \phi_2^+)$ of the Zakharov–Shabat equation also be a solution for $k \in \mathbb{R}$ of

$$\begin{aligned} &\mu_t^+ = V^+\mu^+ + \mu^+ e^{-ik\sigma_3 x}\Omega^+ e^{ik\sigma_3 x}, \\ &V^+ = \frac{1}{4i}\int \frac{g(\lambda)d\lambda}{\lambda - (k+i0)} \begin{pmatrix} |a|^2 - |b|^2 & 2a\overline{b}e^{-2i\lambda x} \\ 2\overline{a}be^{2i\lambda x} & |b|^2 - |a|^2 \end{pmatrix}. \end{aligned} \tag{4.4}$$

We note that because the matrix $V(k,x,t)$ is discontinuous when k crosses the real axis, we must write the time evolution in a given half-plane. A similar equation could be written for $\mu^- = (\phi_1^-, \psi_2^-)$ with V^- and Ω^-, but it is redundant because of the reduction symmetry. The compatibility between the two operators in Eqs. (2.1) and (2.4) with the constraint $\Omega_x = 0$ results in Eq. (4.1) within reduction (2.33).

4.2. Evolution of the spectral transform. The free entry $\Omega^+ = \Omega^+(k,t)$ is the *dispersion relation* [15] because it coincides with the dispersion law of the linearized evolution in the infinite line case. Differential equation (4.4) is evaluated at $x = 0$ and $x \to \infty$, which gives eight equations for the four components of Ω^+ and the four evolution equations for ρ, τ, and their conjugates (these evolutions are compatible, which actually results from the reduction). We obtain

$$\Omega^+(k,t) = -\frac{i}{4}\mathcal{I}(k)\left[2\rho(k,t)\bar{A}B\sigma_3 + \begin{pmatrix} |A|^2 - |B|^2 & 0 \\ 2\bar{A}B & -|A|^2 + |B|^2 \end{pmatrix}\right], \tag{4.5}$$

where we define $\mathcal{I}(k)$ as

$$\mathcal{I}(k) = \int \frac{g(\lambda)d\lambda}{\lambda - k}, \quad \mathrm{Im}(k) > 0 \tag{4.6}$$

(we note that the dispersion relation depends on the spectral transform itself), and the Riccati evolution

$$\rho_t = \frac{i}{2}\mathcal{I}(k)\big[\bar{A}B\rho^2 - (|A|^2 - |B|^2)\rho - A\overline{B}\big]. \tag{4.7}$$

It was proved in [11] that the above evolution preserves the large-k expansion of $\rho(k,0)$.

4.3. The sine-Gordon limit. The sine-Gordon equation is obtained from SRS system (4.1) when, first, the distribution $g(k)$ goes to the Dirac distribution, namely,

$$g(k) = -\delta(k), \tag{4.8}$$

and, second, the boundary data are real and normalized, i.e.,

$$A, B, Q \in \mathbb{R}, \qquad A^2 + B^2 = 1. \tag{4.9}$$

We note that from the conservation law $\partial_x(|a|^2 + |b|^2) = 0$, this normalization is not restrictive. It is then easy to demonstrate directly from (4.1) that the fields remain real-valued as time evolves. The final mapping to the sine-Gordon equation is achieved by setting

$$a = \cos\frac{\theta}{2}, \qquad b = \sin\frac{\theta}{2}, \tag{4.10}$$

which maps (4.1) to

$$\theta_{xt} + \sin\theta = 0, \qquad q = -\frac{1}{2}\theta_x. \tag{4.11}$$

The boundary data are then related through

$$A = \cos\frac{\theta_0}{2}, \qquad B = \sin\frac{\theta_0}{2}, \tag{4.12}$$

and the evolution of the spectral transform in Eq. (4.7) is precisely mapped onto evolution (3.6) because with choice (4.8) of the delta function for $g(k)$, the integral becomes $\mathcal{I}(k) = 1/k$.

We can now use this equivalence to consider the sine-Gordon equation as a *limit* by choosing the Lorentzian

$$g(k) = -\frac{1}{\pi}\frac{\epsilon}{k^2 + \epsilon^2} \tag{4.13}$$

with $\epsilon > 0$. Because the integral $\mathcal{I}(k)$ is written for $\mathrm{Im}(k) > 0$, we can then use contour integration in the lower half-plane to obtain

$$\mathcal{I}(k) = \frac{1}{k + i\epsilon}. \tag{4.14}$$

Consequently, the time evolution of the spectral transform with boundary values (4.12) is given by

$$\rho_t = \frac{i}{4(k + i\epsilon)}\big[\rho^2 \sin\theta_0 - 2\rho\cos\theta_0 - \sin\theta_0\big]. \tag{4.15}$$

The limit as $\epsilon \to 0^+$ leads to the sine-Gordon equation, and the above evolution regularizes the spectral transform.

4.4. Application. The application can be illustrated in the simple, but representative, case where θ_0 is a one-step function

$$\theta_0(t) = 0, \quad t \in [0, t_1], \qquad \theta_0(t) = \varphi_1, \quad t > t_1, \tag{4.16}$$

and the medium is initially at rest, i.e., $\theta(x, 0) = 0$, which, from the spectral transform standpoint means that $\rho(k, 0) = 0$. From the preceding section, the solution $\rho(k, t)$ is then given by

$$\rho(k,t) = (1 - \cos\varphi_1)\left[1 - e^{i(t-t_1)(2(k+i\epsilon))}\right]\left[e^{i(t-t_1)(2(k+i\epsilon))} + \frac{1 - \cos\varphi_1}{1 + \cos\varphi_1}\right]^{-1}. \tag{4.17}$$

The poles $k_n(t) = \zeta_n + i\eta_n$ of ρ are then the solutions of

$$\zeta_n = 2(n+1)\frac{\pi}{\alpha}(\eta_n + \epsilon), \qquad \eta_n = \frac{\alpha(t - t_1)}{\alpha^2 + 4(n+1)^2\pi^2} - \epsilon, \tag{4.18}$$

where we assume that $\varphi_1 \in [0, \pi]$ and define $\alpha = \log(1 - \cos\varphi_1) - \log(1 + \cos\varphi_1)$. We note that these poles occur in pairs $\{k_n, k_m = -\bar{k}_n\}$, because

$$m = -n - 2 \quad \Rightarrow \quad \{\eta_m = \eta_n,\ \zeta_m = -\zeta_n\}. \tag{4.19}$$

The constant ϵ regularizes the spectral transform as follows. At the initial time $t = t_1$, all poles lie on the essential singularity $-i\epsilon$, which would belong to the real axis in the sine-Gordon case $\epsilon = 0$. If $\varphi_1 < \pi/2$, all poles remain in the lower half-plane, and the spectral transform has no discrete part. But if $\varphi_1 > \pi/2$, then as time runs, the poles move from the point $-i\epsilon$ and cross the real axis (in pairs) at the *successive* times T_n,

$$T_n = t_1 + \frac{\epsilon}{\alpha}[\alpha^2 + 4(n+1)^2\pi^2]. \tag{4.20}$$

For $\epsilon = 0$, all poles (an infinite number) would therefore cross the real axis at *the same time* t_1.

One of the remaining open problems is the identification of the breather in the solution $\theta(x, t)$ itself. The problem is that in contrast to the infinite-line case, the spectrum cannot contain *only* the discrete part, and the reconstruction of the solution from the spectrum is therefore *not explicit*. This problem is inherent in boundary value problems.

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