A mixed boundary qKZ equation: integrability, graphical solutions, and connections

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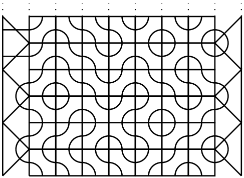
Outline

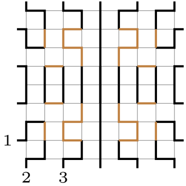
1. Temperley–Lieb loop model
2. Solutions of the qKZ equation
3. Hecke algebra and special functions

Outline

Temperley–Lieb loop model



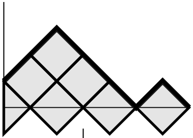
Fully packed loops



Razumov–Stroganov conjectures

Stochastic process
$M|\Psi_0\rangle = 0$

Ballot paths



qKZ solutions
Hecke algebras

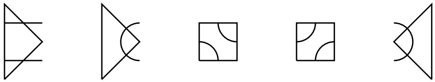
Section 1

Temperley–Lieb loop model

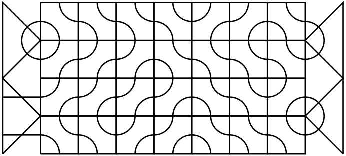
Temperley–Lieb $O(n)$ loop model

Random tiling with

- Tiles: left boundary, bulk, right boundary



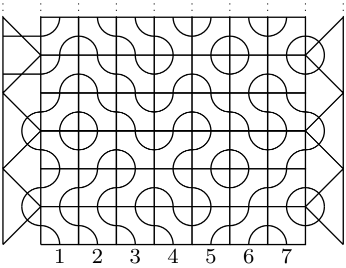
- Mixed boundaries: open or reflecting at left, always reflecting at right
- An example configuration with four rows:



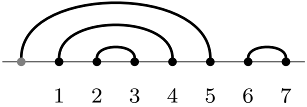
- Closed loops are given weight $n = -(t + t^{-1})$
- Choosing $t = \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$ gives loops weight $n = 1$

Link patterns

- Consider now a semi-infinite lattice, width N

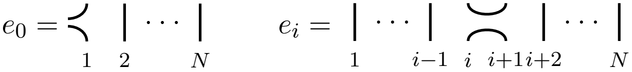


- Represent connectivity along the bottom edge by left-extended link patterns

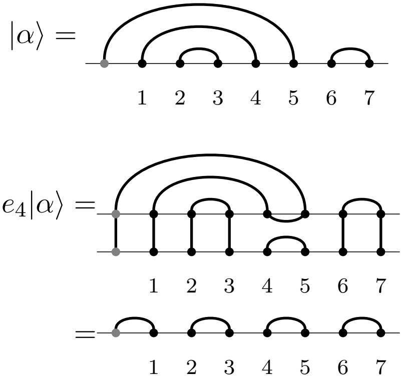


Action on link patterns

- Introduce operators $e_0, e_1, \ldots, e_{N-1}$



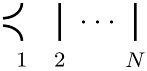
- Act on a link pattern

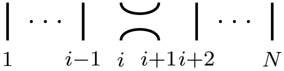


Temperley–Lieb algebra

Introduce the one boundary Temperley–Lieb algebra

- The operators e_0, $e_1, \ldots, e_{N-1}$ are the generators

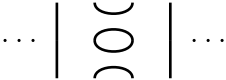
$e_0 =$ 

$e_i =$ 

- Relations

$$e_i^2 = -(t+t^{-1})e_i, \quad e_i e_{i\pm1} e_i = e_i, \quad e_i e_j = e_j e_i, |i-j| > 1,$$
$$e_0^2 = e_0, e_1 e_0 e_1 = e_1$$

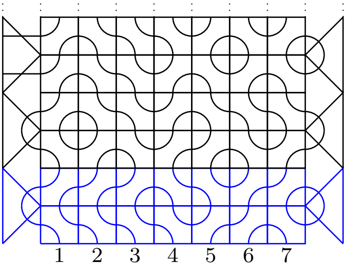
- Example

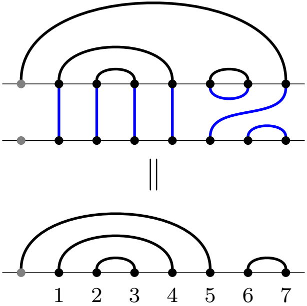
$e_i^2 =$  $= -(t+t^{-1})$ 

$$= -(t+t^{-1})e_i$$

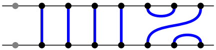
Adding rows

- Adding a pair of rows transforms the link pattern





- In terms of the Temperley–Lieb generators

 $= e_5 e_6$

Transfer matrix

- The *double row transfer matrix* gives the probability of transitions between link patterns
- Give weights to tiles, e.g.

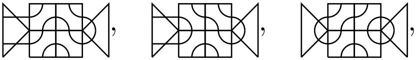
 $\sim a(w)$  $\sim b(w)$

- For $N = 2$

$$\boldsymbol{t}(w) = \begin{array}{c} \\ |\Omega\rangle \\ |\alpha\rangle \end{array} \begin{array}{c} \begin{array}{cc} |\Omega\rangle & |\alpha\rangle \end{array} \\ \begin{pmatrix} & \\ * & \end{pmatrix} \end{array}$$

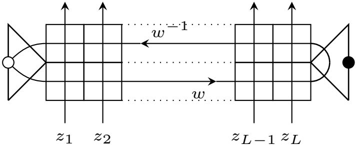
$|\Omega\rangle =$  $|\alpha\rangle =$ 

Contributions from

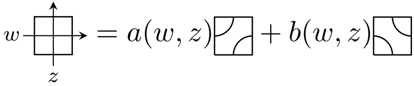
 $\ldots$

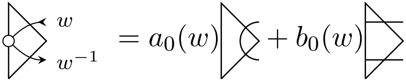
Transfer matrix

- The general case defined pictorially

$\boldsymbol{t}(w; z_i) =$ 

- Assigns weights to tiles





- Expanding gives a weighted sum over all double row configurations

Integrability

- Choose the weights so that the model is *integrable*

$$[\boldsymbol{t}(w;z_i),\boldsymbol{t}(v;z_i)]=0$$

- Then the eigenvectors are independent of the spectral parameter

$$\boldsymbol{t}(w;z_i)|\Psi(z_i)\rangle=\Lambda(w)|\Psi(z_i)\rangle$$

- Integrability in this model arises from solutions of the Yang–Baxter and reflection relations

Yang–Baxter and reflection relations

- Yang–Baxter relation

$$R_i(w)R_{i+1}(wz)R_i(z) = R_{i+1}(z)R_i(wz)R_{i+1}(w)$$

- Reflection relation

$$K_0(z)R_1(wz)K_0(w)R_1(w/z) = R_1(w/z)K_0(w)R_1(wz)K_0(z)$$

- Satisfied by the bulk R matrix, and boundary K matrix

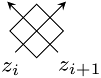
$$R_i(z) = \frac{t - t^{-1}z}{tz - t^{-1}}\mathbb{1} - \frac{z-1}{tz - t^{-1}}e_i$$

$$K_0(z) = \frac{(1 - z^{-1}\zeta_1^{-1})(z - t\zeta_1)}{(z-\zeta_1)(t - z^{-1}\zeta_1^{-1})}\mathbb{1} - \frac{(1-t)(z - z^{-1})}{(z-\zeta_1)(t - z^{-1}\zeta_1^{-1})}e_0$$

- The transfer matrix weights are related to the coefficients in these expressions

Interlacing condition

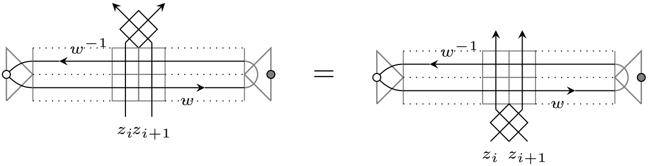
- The R operator is represented graphically as

$R_i(z_i/z_{i+1}) =$ 

- We will need the bulk interlacing condition

$$R_i\left(\frac{z_i}{z_{i+1}}\right)\boldsymbol{t}(w;\ldots,z_i,z_{i+1},\ldots) = \boldsymbol{t}(w;\ldots,z_{i+1},z_i,\ldots)R_i\left(\frac{z_i}{z_{i+1}}\right)$$

- Or pictorially



The transition matrix

- Taking $-(t+t^{-1})=1$ we can obtain a stochastic transition matrix

$$M = \alpha \frac{\partial}{\partial w} \log \boldsymbol{t}(w; z_i = 1)\Big|_{w=1} + \text{const.}$$

where

$$M = a(e_0 - \mathbb{1}) + \sum_{i=1}^{L-1} (e_i - \mathbb{1}), \qquad [M, t(u)] = 0$$

- M has left eigenvector $(1, \dots, 1)$ with eigenvalue 0. Perron–Frobenius tells us the corresponding right eigenvector is the unique stationary state.
- Will assume there is a transfer matrix eigenvector such that

$$\boldsymbol{t}(w; z_i)|\Psi_0(z_i)\rangle = |\Psi_0(z_i)\rangle$$

then

$$M|\Psi_0(z_i = 1)\rangle = 0$$

so $|\Psi_0(z_i)\rangle$ is unique (in the neighbourhood of $z_i = 1$)

The qKZ equations

- Using the definition of the stationary state, and the interlacing condition

$$\begin{aligned} R_i(z_i/z_{i+1})|\Psi_0(z_i)\rangle &= R_i(z_i/z_{i+1})t(w;z_i,z_{i+1})|\Psi_0(z_i)\rangle \\ &= t(w;z_{i+1},z_i)R_i(z_i/z_{i+1})|\Psi_0(z_i)\rangle \end{aligned}$$

- Then from uniqueness can show that

$$R_i(z_i/z_{i+1})|\Psi_0(\ldots,z_i,z_{i+1},\ldots)\rangle = |\Psi_0(\ldots,z_{i+1},z_i,\ldots)\rangle$$

  This is the bulk part of the qKZ equation.

- The boundary equations

$$\begin{aligned} K_0(z_1^{-1})|\Psi(z_1,z_2,\ldots,z_N)\rangle &= |\Psi(z_1^{-1},z_2,\ldots,z_N)\rangle, \\ |\Psi(z_1,\ldots,z_{N-1},z_N)\rangle &= |\Psi(z_1,\ldots,z_{N-1},t^3 z_N^{-1})\rangle \end{aligned}$$

Summary so far

- We have a stochastic process

$$M = a(e_0 - \mathbb{1}) + \sum_{i=1}^{L-1} (e_i - \mathbb{1})$$

  with stationary distribution $|\Psi_0\rangle$ such that $M|\Psi_0\rangle = 0$
- To find $|\Psi_0\rangle$ we will find the more general vector

$$|\Psi_0\rangle \to |\Psi_0(z_1, \ldots z_N)\rangle$$

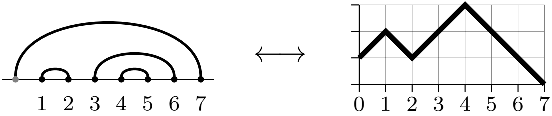
  by solving the qKZ equations
- Setting $z_i = 1$ will give us back the stationary distribution. But we are also interested in the general solution!

Section 2

Solutions of the qKZ equation

Bijection to Ballot paths

- Left-extended link patterns are in bijection with Ballot paths



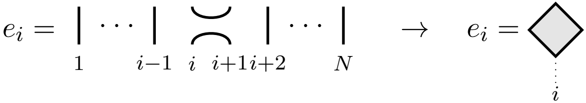
- Working from right to left:
  - Arch opening - step left and up
  - Arch closing - step left and down
- Gives a Ballot path: sequence of non-negative heights

$$\alpha = (\alpha_0, \ldots, \alpha_N)$$

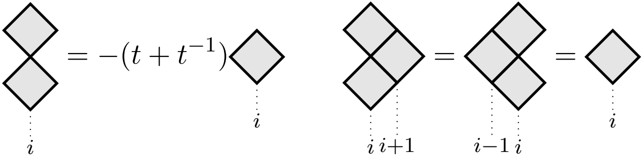
with $\alpha_N = 0$, and $\alpha_{i+1} - \alpha_i = \pm 1$.

The Temperley–Lieb algebra

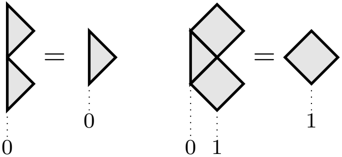
- Generators mapped to tiles



- Bulk relations $e_i^2 = -(t + t^{-1})e_i$, $e_i e_{i\pm 1} e_i = e_i$:



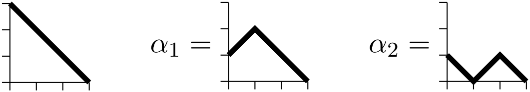
- Boundary relations $e_0^2 = e_0$, $e_1 e_0 e_1 = e_1$:



Action on Ballot paths

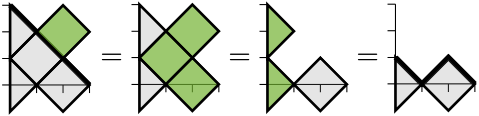
- Ballot paths of length $N = 3$

$\Omega =$, $\alpha_1 =$, $\alpha_2 =$



- Example

$e_2|\Omega\rangle = \; = \; = \; = |\alpha_2\rangle$



- Matrix form

$$e_2 = \begin{array}{c} \\ |\Omega\rangle \\ |\alpha_1\rangle \\ |\alpha_2\rangle \end{array} \begin{array}{c} \begin{array}{ccc} |\Omega\rangle & |\alpha_1\rangle & |\alpha_2\rangle \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -(t+t^{-1}) \end{pmatrix} \end{array}$$

Components of the qKZ equation

- Write the stationary state vector in the Ballot path basis as

$$|\Psi(z_1,\ldots,z_N)\rangle = \sum_\alpha \psi_\alpha(z_1,\ldots,z_N)|\alpha\rangle$$

- The bulk part of the qKZ equation

$$R_i(z_i/z_{i+1})|\Psi(z_1,\ldots,z_N)\rangle = |\Psi(\ldots,z_{i+1},z_i,\ldots)\rangle$$

- Component form

$$\sum_\alpha \psi_\alpha(z_1,\ldots,z_N)\Big(e_i|\alpha\rangle\Big) = \sum_\alpha \Big(T_i(-1)\psi_\alpha(z_1,\ldots,z_N)\Big)e_i|\alpha\rangle$$

where the $T_i(u)$ are Hecke operator acting on Laurent polynomials

- The boundary equations

$$K_0(z_1^{-1})|\Psi(z_1,z_2,\ldots,z_N)\rangle = |\Psi(z_1^{-1},z_2,\ldots,z_N)\rangle,$$

$$|\Psi(z_1,\ldots,z_{N-1},z_N)\rangle = |\Psi(z_1,\ldots,z_{N-1},t^3 z_N^{-1})\rangle$$

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Solution of the qKZ equation

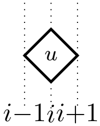
Theorem (de Gier, Pyatov 2010)

The solutions of the qKZ equation have a factorised form

$$\psi_\alpha(z_1,\ldots,z_N) = \prod_{i,j}^{\nearrow u_{i,j}} T_i(u_{i,j})\psi_\Omega(z_1,\ldots,z_N)$$

The product is constructed using a graphical representation of the Hecke generators

$T_0(u) =$ 

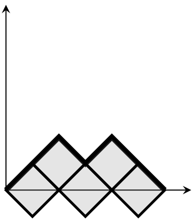
, $\qquad T_i(u) =$ 

.

These are operators on Laurent polynomials, which also satisfy Yang–Baxter and reflection relations.

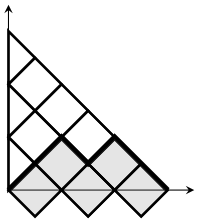
Factorised solutions

- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$



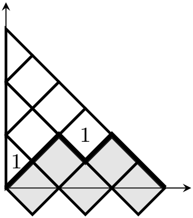
Factorised solutions

- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$
- Fill to maximal Ballot path $\Omega = (N, N-1, \ldots, 0)$



Factorised solutions

- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$
- Fill to maximal Ballot path $\Omega = (N, N-1, \ldots, 0)$
- Label corners with 1



Factorised solutions

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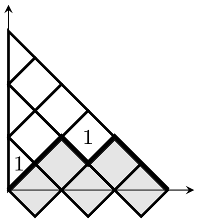
- Label remaining tiles by rule $u_{i,j} = \max\{u_{i+1,j-1}, u_{i-1,j-1}\} + 1$



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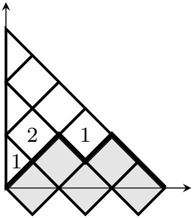

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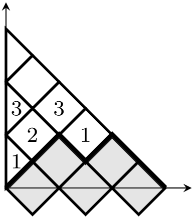

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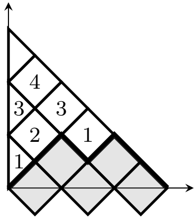

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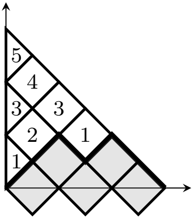

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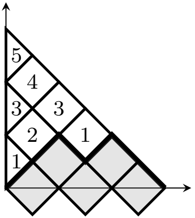


$$\psi_\alpha = T_0(1).T_1(2)T_0(3).T_3(1)T_2(3)T_1(4)T_0(5)\psi_\Omega$$

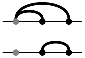
and

$$\psi_\Omega = \Delta_t^-(z_1, \ldots, z_N)\Delta_t^+(z_1, \ldots, z_N)$$

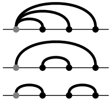
Stationary state solutions

- The stationary state can be calculated directly from the factorised solutions

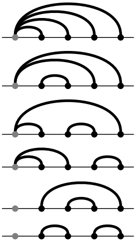
$$|\Psi_0^{(2)}\rangle = \frac{1}{\mathcal{Z}_2}\begin{pmatrix}1\\1\end{pmatrix}$$



$$|\Psi_0^{(3)}\rangle = \frac{1}{\mathcal{Z}_3}\begin{pmatrix}1\\2\\3\end{pmatrix}$$



$$|\Psi_0^{(4)}\rangle = \frac{1}{\mathcal{Z}_4}\begin{pmatrix}1\\3\\8\\9\\3\\9\end{pmatrix}$$



for $\zeta_0 = t$, $t = \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$.

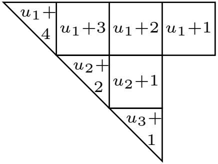
- Will return to the integer entries later
- Computing the entries for large N is difficult (no closed form)

Alternate filling

Fill with consecutive integers along rows, e.g. for previous shape tilted by 45 °

$$\psi_{4,2,1}(u_1+1, u_2+1, u_3+1)$$

$$=$$



$$= \mathcal{T}_1(u_3+1)\mathcal{T}_2(u_2+1)\mathcal{T}_3(u_1+1)\psi_\Omega$$

where

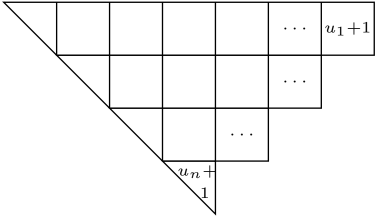
$$\mathcal{T}_a(u+1) = T_{a-1}(u+1)\ldots T_1(u+a-1)T_0(u+a)$$

gives a row of length a, numbered from $u+1$.

Staircase diagram

- Call the largest such element the *staircase diagram*:

$$\psi_{\bar{a}_1,\dots,\bar{a}_n}(u_1+1,\dots,u_n+1) =$$



where $n = \lfloor N/2 \rfloor$, $\bar{a}_i = N - 2i + 1$

- In terms of Hecke generators

$$\psi_{\bar{a}_1,\dots,\bar{a}_n}(u_1+1,\dots,u_n+1)$$
$$= \mathcal{T}_{N-2n+1}(u_n+1)\dots\mathcal{T}_{N-3}(u_2+1)\mathcal{T}_{N-1}(u_1+1)\psi_\Omega$$

Generalised sum rule

Theorem (de Gier, F)

The staircase diagram has the expansion

$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1) = \sum_\alpha c_\alpha \psi_\alpha(z_1,\ldots,z_N),$$

where the coefficients c_α are non-zero and are monomials in

$$y_i = -\frac{[u_i]}{[u_i+1]}, \qquad \tilde{y}_i = -B_0(u_i+1).$$

- Using the notation

$$[u] = [u]_t = \frac{t^u - t^{-u}}{t - t^{-1}}$$

- Proof of the sum rule requires expanding staircase diagram in two stages.

First expansion

The first stage of the expansion gives the form of the coefficients.

Lemma (First expansion)

$$\mathcal{T}_{a_n}(u_n+1)\dots\mathcal{T}_{a_1}(u_1+1)\psi_\Omega$$
$$= \prod_{i=n,n-1,\dots,1} \left(\mathcal{T}_{a_i}(1) + y_i \mathcal{T}_{a_i-1}(1) + \tilde{y}_i\right)\psi_\Omega$$

where

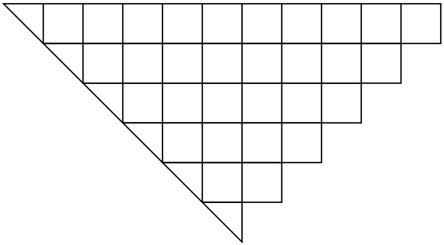
$$y_i = -\frac{[u_i]}{[u_i+1]}, \qquad \tilde{y}_i = -B_0(u_i+1)$$

First expansion terms

Procedure to expand

$$(\mathcal{T}_{\bar{a}_n}(1) + y_n \mathcal{T}_{\bar{a}_n - 1}(1) + \tilde{y}_n) \ldots (\mathcal{T}_{\bar{a}_1}(1) + y_1 \mathcal{T}_{\bar{a}_1 - 1}(1) + \tilde{y}_1)\, \psi_\Omega$$

- Start from the empty outline.
- Working from top down, a row may be left empty (factor $\tilde{y}_i$), filled one short (factor y_i), or filled completely (no additional factor).
- Delete empty rows and boxes.



First expansion terms

Procedure to expand

$$(\mathcal{T}_{\bar{a}_n}(1) + y_n \mathcal{T}_{\bar{a}_n - 1}(1) + \tilde{y}_n) \dots (\mathcal{T}_{\bar{a}_1}(1) + y_1 \mathcal{T}_{\bar{a}_1 - 1}(1) + \tilde{y}_1)\, \psi_\Omega$$

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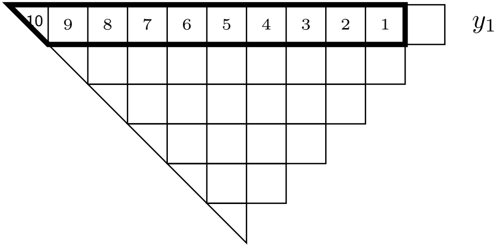

First expansion terms

Procedure to expand

$$(\mathcal{T}_{\bar{a}_n}(1) + y_n \mathcal{T}_{\bar{a}_n - 1}(1) + \tilde{y}_n) \ldots (\mathcal{T}_{\bar{a}_1}(1) + y_1 \mathcal{T}_{\bar{a}_1 - 1}(1) + \tilde{y}_1)\, \psi_\Omega$$

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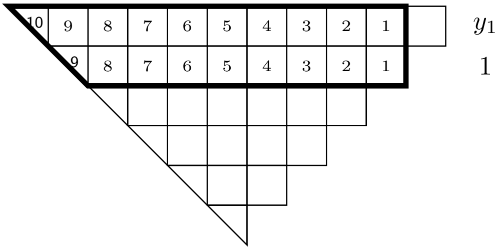

First expansion terms

Procedure to expand

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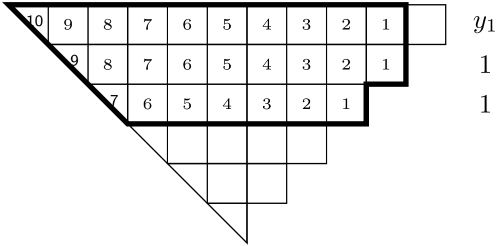

First expansion terms

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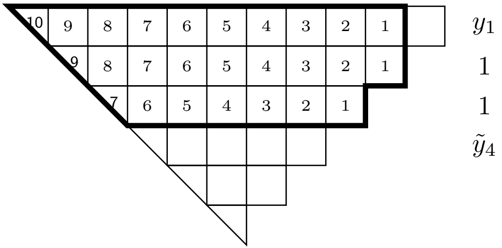

First expansion terms

Procedure to expand

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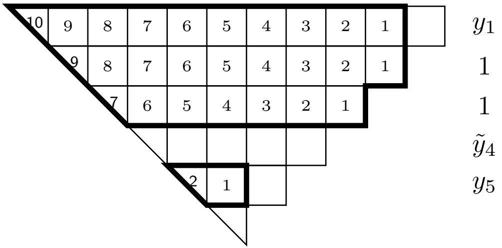

First expansion terms

Procedure to expand

$$\left(\mathcal{T}_{\bar{a}_n}(1) + y_n \mathcal{T}_{\bar{a}_n - 1}(1) + \tilde{y}_n\right) \ldots \left(\mathcal{T}_{\bar{a}_1}(1) + y_1 \mathcal{T}_{\bar{a}_1 - 1}(1) + \tilde{y}_1\right) \psi_\Omega$$

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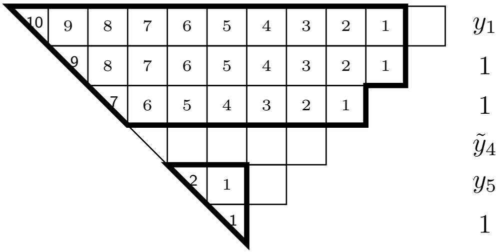

First expansion terms

Procedure to expand

$$\left(\mathcal{T}_{\bar{a}_n}(1) + y_n \mathcal{T}_{\bar{a}_n - 1}(1) + \tilde{y}_n\right) \dots \left(\mathcal{T}_{\bar{a}_1}(1) + y_1 \mathcal{T}_{\bar{a}_1 - 1}(1) + \tilde{y}_1\right) \psi_\Omega$$

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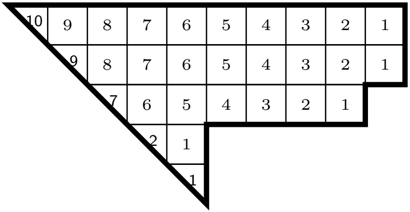


- Coefficient $y_1 \tilde{y}_4 y_5$

Second expansion

When the resulting term is not a proper component ψ_α, a second expansion is required.

Lemma (Second expansion)

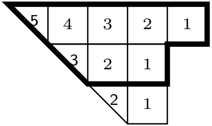
Let $\psi_\alpha(z_1, \ldots, z_N)$ be a component of the qKZ solution, with last row of length $a+1$, then

$$T_{a-1}(1) \ldots T_1(a-1) T_0(a) \psi_\alpha(z_1, \ldots, z_N) = \sum_{\alpha'} \psi_{\alpha'}(z_1, \ldots, z_N)$$

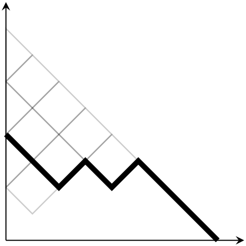
- The terms in the sum are found through a graphical rule, and all have coefficient 1.

Second expansion example

$$T_1(1)T_0(2)\psi_\alpha(z_1,\ldots,z_N) =$$



Ballot path



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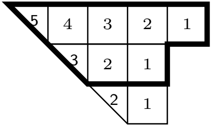
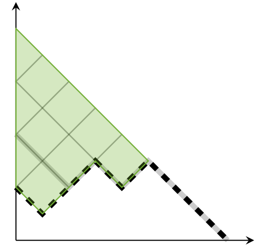
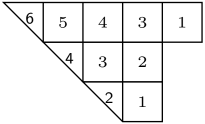


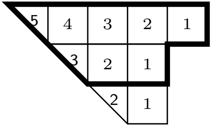
Ballot path



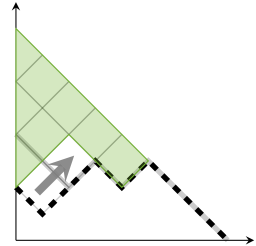
Terms



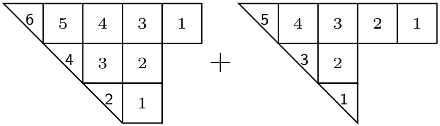
Second expansion example

$T_1(1)T_0(2)\psi_\alpha(z_1, \ldots, z_N) =$ 

Ballot path



Terms



Second expansion example

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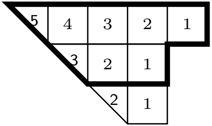
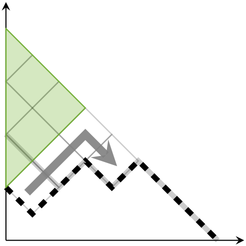


Ballot path



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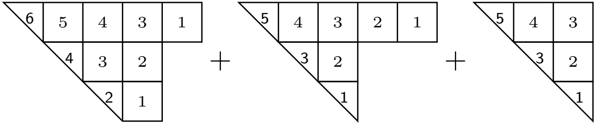


Proof of the sum rule

- Recall the sum rule

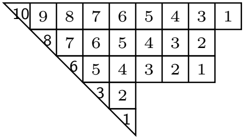
$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1) = \sum_\alpha c_\alpha \psi_\alpha(z_1,\ldots,z_N),$$

  where the coefficients c_α are non-zero and are monomials in y_i, $\tilde{y}_i$.
- We have shown via the two expansions that the staircase diagram can be expanded in terms of components ψ_α, with coefficients polynomials in y_i, $\tilde{y}_i$.
- To show that the coefficients are non-zero and monomials, we must show that each component ψ_α arises from a single term in the first expansion.

Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



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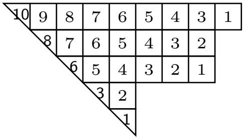
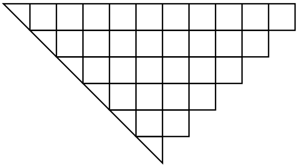


- Draw empty maximal staircase



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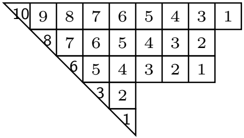
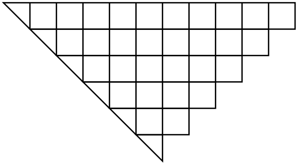


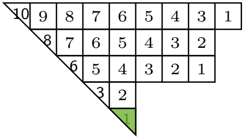
- Draw empty maximal staircase
- Add rows to staircase, bottom up, in lowest place each fits



Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



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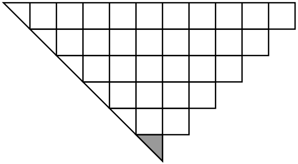

Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

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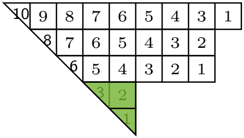

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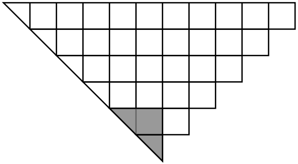
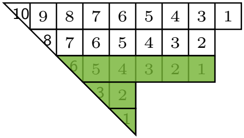


Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



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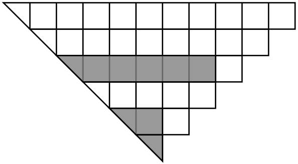

Example of the algorithm

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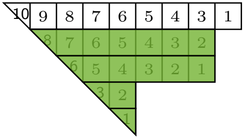

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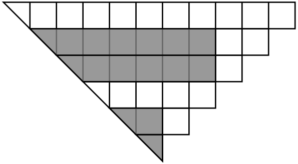

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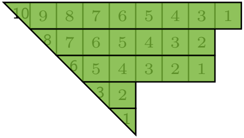

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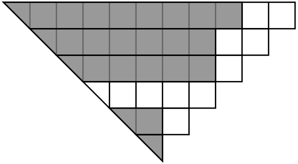

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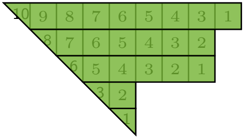
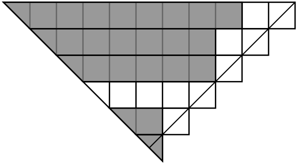
$\psi_\alpha(z_1, \ldots, z_N) =$ 

- Draw empty maximal staircase
- Add rows to staircase, bottom up, in lowest place each fits
- Draw in ribbons, starting from outer diagonal



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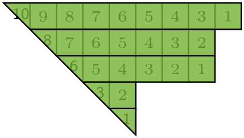
$\psi_\alpha(z_1, \ldots, z_N) =$ 

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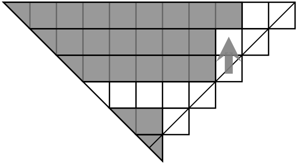

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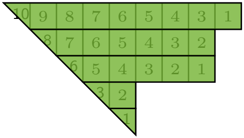

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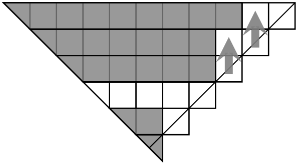

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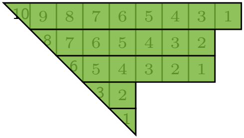
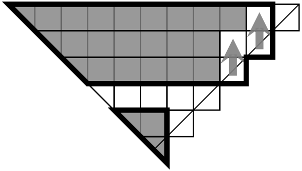
- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



- Draw empty maximal staircase
- Add rows to staircase, bottom up, in lowest place each fits
- Draw in ribbons, starting from outer diagonal



- Coefficient $c_\alpha = y_1 \tilde{y}_4 y_5$.

Specialisation of the sum rule

- At specialisation $u_i = 1$, $t = \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$, all coefficients $c_\alpha = 1$.

$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(2,\ldots,2) = \sum_\alpha \psi_\alpha(z_1,\ldots,z_N),$$

  giving the normalisation of the loop model stationary state vector

- At this point there is a closed form for the sum [Zinn-Justin 2007]. Setting $z_i = 1$ and $\zeta_1 = t = \mathrm{e}^{\pm 2\pi \mathrm{i}/3}$

$$\mathcal{Z}_N = \sum_\alpha \psi_\alpha(z_i = 1) = \prod_{k=1}^{N} \frac{\lfloor 3k/2 + 1 \rfloor (3k)! k!}{(2k+1)!(2k)!}$$

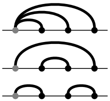
  giving

$$\mathcal{Z}_2 = 2, \quad \mathcal{Z}_3 = 6, \quad \mathcal{Z}_4 = 33 \ldots$$

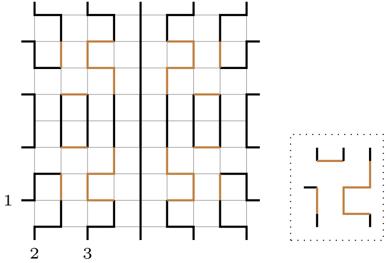
Razumov–Stroganov conjectures

- The sequence $\mathcal{Z}_N$ counts the number of vertically and horizontally symmetric fully packed loop diagrams (FPLs) of size $2N+3$.

$$|\Psi_0^{(3)}\rangle = \frac{1}{\mathcal{Z}_3}\begin{pmatrix}1\\2\\3\end{pmatrix}$$



$$\overset{?}{\leftrightarrow}$$



Fully packed loops

- This connection between Temperley–Lieb loop models and FPLs is one of several Razumov–Stroganov type conjectures.

Section 3

Hecke algebra and special functions

Hecke algebra

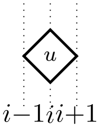
The operators $T_i(u)$ are polynomial representations of a Baxterized Hecke algebra.

- Hecke algebra, $\mathcal{H}$, with relations

$$(T_i - t)(T_i + t^{-1}) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad i \geq 1$$
$$T_i T_j = T_j T_i, \quad \forall i, j : |i - j| > 1$$

- Baxterized generator

$T_i(u) =$

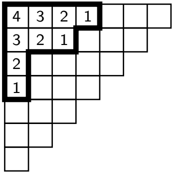


$$= T_i + t^{-1} - \frac{[u-1]}{[u]}$$

- We have also seen the boundary element T_0 but for the moment we will consider *periodic* systems (type A) without this generator.

qKZ solutions for type A

- Type A solutions given by partitions labelled with the same rule as the mixed boundary system [Kirilov, Lascoux 2000, de Gier, Pyatov 2010], e.g.



- The type A base function has form

$$\psi_{\Omega}^{(A)}(z_1, \ldots, z_2 n) = \Delta(z_1, \ldots, z_n)\Delta(z_{n+1}, \ldots, z_{2n+1})$$

- The set of solutions forms the Kazhdan–Lusztig basis

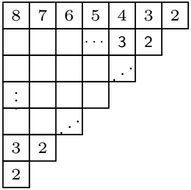
$$\mathcal{H}\Delta\Delta = \text{span}\{\psi_{\alpha}^{(A)}\}$$

with invariance property

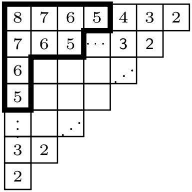
$$\overline{\psi_{\alpha}^{(A)}} = \psi_{\alpha}^{(A)}$$

Sum rule for type A

- Sum rule given by consecutive integer labelling [de Gier, Lascoux, Sorrell 2012]



- Set of all subpartitions gives the Young basis, e.g.



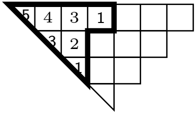
- Elements of the Young basis are specialised Macdonald polynomials.

Macdonald polynomials

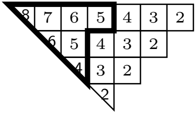
- Within the Hecke algebra, can define a family of commuting elements
$$Y_i = T_i \dots T_{N-1} \omega T_1^{-1} \dots T_{i-1}^{-1}$$
with
$$[Y_i, Y_j] = 0$$
- These operators have a shared set of eigenfunctions E_λ, with
$$Y_i E_\lambda(z_1, \dots, z_N) = y(\lambda)_i E_\lambda(z_1, \dots, z_N)$$
and these E_λ are the non-symmetric Macdonald polynomials
- The eigenfunctions are related by intertwiners
$$E_{s_i\lambda}(z_1, \dots, z_N) = T_i(u(\lambda)_i) E_\lambda(z_1, \dots, z_N)$$
- For *periodic* boundaries, the intertwining relation gives exactly the Young basis elements

Hecke bases for mixed boundaries

- The elements of the qKZ solution correspond to a Kazhdan–Lusztig basis for the mixed boundary (type B) Hecke algebra [Shigechi 2014], e.g.



- The consecutive integer numbering gives an alternative basis, e.g.



- The expansion rules that led to the sum rule give the change of basis back to the KL basis
- Koornwinder instead of Macdonald polynomials

Conclusion and prospects

- The Temperley–Lieb loop model connects several areas of mathematics - integrability, combinatorics, representation theory, ...
- Though the stationary state is given through solutions of the qKZ equation we do not have a closed form
- The construction of the generalised sum rule gives a change of basis of the Hecke algebra, and relates Koornwinder polynomials to the qKZ solution
- We are hopeful that this will help us find closed forms for elements of the loop model stationary distribution